Exercise 9.1

Determine order and degree (if defined) of differential equations given in Exercise 1 to 10:

1. $\frac{d^4y}{dx^4}$ **+ sin (y''') = 0**

Sol. The given D.E. is $\frac{a^4y}{az^4} + \sin y''' = 0$

The highest order derivative present in the differential equation is $\frac{a^4y}{az^4}$ and its order is 4.

The given differential equation is not a polynomial equation in derivatives (∵ The term sin y''' is a T-function of derivative y'''). Therefore degree of this D.E. is not defined.

Ans. Order 4 and degree not defined.

2. $y' + 5y = 0$

Sol. The given D.E. is $y' + 5y = 0$.

The highest order derivative present in the D.E. is $y' \left(= \frac{ay}{az} \right)$ and so its order is one. The given D.E. is a polynomial equation in derivatives (y' here) and the highest power raised to highest order derivative y' is one, so its degree is one.

Ans. Order 1 and **degree 1.**

3. $\left(\frac{ds}{d5}\right)^4 + 3s\frac{d^2s}{d5^2} = 0$

Sol. The given D.E. is $\left(\frac{az}{at}\right)^4 + 3s\frac{a^2z}{at^2} = 0.$

The highest order derivative present in the D.E. is $\frac{a^2z}{at^2}$ and its order is 2. The given D.E is a polynomial equation in derivatives and the highest power raised to highest order derivative $\frac{a^2z}{at^2}$ is one. Therefore degree of D.E. is 1.

Ans. Order 2 and degree 1.

4. $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\frac{dy}{dx} = 0$

Sol. The given D.E. is $\left(\frac{a^2y}{az^2}\right)^2 + \cos\left(\frac{ay}{az}\right) = 0.$

The highest order derivative present in the differential equation is $\frac{a^2y}{az^2}$ and its order is 2.

The given D.E. is not a polynomial equation in derivatives ($\because$ The term $\cos\frac{ay}{az}$ is a T-function of derivative $\frac{ay}{az}$).

Therefore degree of this D.E. is not defined.

Ans. Order 2 and degree not defined.

5. $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$

Sol. The given D.E. is $\frac{a^2y}{az^2} = \cos 3x + \sin 3x.$

The highest order derivative present in the D.E. is $\frac{a^2y}{az^2}$ and its order is 2.

The given D.E. is a polynomial equation in derivatives and the highest power raised to highest order $\frac{a^2y}{az^2} = \left(\frac{a^2y}{az^2}\right)^1$ is one, so its degree is 1.

Ans. Order 2 and degree 1.

Remark. It may be remarked that the terms cos 3*x* and sin 3*x* present in the given D.E. are trigonometrical functions (but not T-functions of derivatives).

It may be noted that $\left(\cos 3\frac{ay}{az}\right)$ is not a polynomial function of derivatives.

6. $\mathbf{(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0}$

Sol. The given D.E. is $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ ....(i)

The highest order derivative present in the D.E. is y''' and its order is 3.

The given D.E. is a polynomial equation in derivatives y''', y'' and y' and the highest power raised to highest order derivative y''' is two, so its degree is 2.

Ans. Order 3 and degree 2.

7. $y''' + 2y'' + y' = 0$

Sol. The given D.E. is $y''' + 2y'' + y' = 0$ ......(*i*)

The highest order derivative present in the D.E. is y''' and its order is 3.

The given D.E. is a polynomial equation in derivatives y''', y'' and y' and the highest power raised to highest order derivative y''' is one, so its degree is 1.

Ans. Order 3 and degree 1.

8. $y' + y = e^x$

Sol. The given D.E. is $y' + y = e^x$ ......(*i*)

The highest order derivative present in the D.E. is y' and its order is 1.

The given D.E. is a polynomial equation in derivative y'. (It may be noted that e^x is an exponential function and not a polynomial function but is not an exponential function of derivatives) and the highest power raised to highest order derivative y' is one, so its degree is 1.

Ans. Order 1 and degree 1.

9. $y'' + (y')^2 + 2y = 0$

Sol. The given D.E. is $y'' + (y')^2 + 2y = 0$ ......(*i*)

The highest order derivative present in the D.E. is y'' and its order is 2.

The given D.E. is a polynomial equation in derivatives y'' and y' and the highest power raised to highest order derivative y'' is one, so its degree is 1.

Ans. Order 2 and degree 1.

10. $y'' + 2y' + \sin y = 0$

Sol. The given D.E. is $y'' + 2y' + \sin y = 0$ ......(*i*)

The highest order derivative present in the D.E. is y'' and its order is 2.

The given D.E. is a polynomial equation in derivatives y'' and y'. (It may be noted that sin y is not a polynomial function of y, it is a T-function of y but is not a T-function of derivatives) and the highest power raised to highest order derivative y'' is one, so its degree is one.

Ans. Order 2 and degree 1.

11. The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0 \text{ is}$$

(A) 3 **(B) 2** **(C) 1** **(D) Not defined.**

Sol. The given D.E. is

$$\left(\frac{a^2y}{az^2}\right)^3 + \left(\frac{ay}{az}\right)^2 + \sin\left(\frac{ay}{az}\right) + 1 = 0 \quad ...(i)$$

This D.E. (*i*) is not a polynomial equation in derivatives.

$$\left[\because \sin\left(\frac{ay}{az}\right) \text{ is a T-function of derivative } \frac{ay}{az}\right]$$

$\therefore$ Degree of D.E. (*i*) is not defined.

Answer. Option (D) is the correct answer.

12. The order of the differential equation

$$2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0 \text{ is}$$

(A) 2 **(B) 1** **(C) 0** **(D) Not defined**

Sol. The given D.E. is $2x^2 \frac{a^2y}{az^2} - 3\frac{ay}{az} + y = 0$

The highest order derivative present in the differential equation is $\frac{a^2y}{az^2}$ and its order is 2.

Answer. Order of the given D.E. is 2.

Exercise 9.2

In each of the Exercises 1 to 6 verify that the given functions (explicit) is a solution of the corresponding differential equation:

1. $y = e^x + 1 : y'' - y' = 0$

Sol. Given: $y = e^x + 1$ ...(*i*)

To prove: y given, by (*i*) is a solution of the D.E. $y'' - y' = 0$ ...(*ii*)

From (*i*), $y' = e^x + 0 = e^x$ and $y'' = e^x$

$\therefore$ L.H.S. of D.E. (*ii*) $= y'' - y' = e^x - e^x = 0$ = R.H.S. of D.E. (*ii*)

$\therefore$ y given by (*i*) is a solution of D.E. (*ii*).

2. $y = x^2 + 2x + C : y' - 2x - 2 = 0$

Sol. Given: $y = x^2 + 2x + C$ ...(*i*)

To prove: y given by (*i*) is a solution of the D.E.

$y' - 2x - 2 = 0$ ...(*ii*)

From (*i*), $y' = 2x + 2$

$\therefore$ L.H.S. of D.E. (*ii*) $= y' - 2x - 2$

$= (2x + 2) - 2x - 2 = 2x + 2 - 2x - 2 = 0$ = R.H.S. of D.E. (*ii*)

$\therefore$ y given by (*i*) is a solution of D.E. (*ii*).

3. $y = \cos x + C : y' + \sin x = 0$

Sol. Given: $y = \cos x + C$ ...(*i*)

To prove: y given by (*i*) is a solution of D.E. $y' + \sin x = 0$ ...(*ii*)

From (*i*), $y' = -\sin x$

$\therefore$ L.H.S. of D.E. (*ii*) $= y' + \sin x = -\sin x + \sin x$

$= 0$ = R.H.S. of D.E. (*ii*)

$\therefore$ y given by (i) is a solution of D.E. (ii).

4. $y = \sqrt{1 + x^2}$ **:** $y' = \dfrac{xy}{1 + x^2}$

Sol. Given: $y = \sqrt{1 + z^2}$ ...(i)

To prove: y given by (i) is a solution of D.E. $y' = \dfrac{zy}{1 + z^2}$ ...(ii)

From (i), $y' = \dfrac{a}{az} \sqrt{1 + z^2} = \dfrac{a}{az} (1 + x^2)^{1/2}$

$= \dfrac{1}{2} (1 + x^2)^{-1/2} \dfrac{a}{az} (1 + x^2) = \dfrac{1}{2} (1 + x^2)^{-1/2} . 2x = \dfrac{z}{\sqrt{1 + z^2}}$ ...(iii)

R.H.S. of D.E. (ii) $= \dfrac{zy}{1 + z^2} = \dfrac{z}{1 + z^2} \sqrt{1 + z^2}$ (By (i))

$= \dfrac{z}{\sqrt{1 + z^2}}$ $\left[\because \dfrac{\sqrt{t}}{t} = \dfrac{\sqrt{t}}{\sqrt{t}\sqrt{t}} = \dfrac{1}{\sqrt{t}}\right]$

$= y'$ [By (iii)] = L.H.S. of D.E. (ii)

$\therefore$ y given by (i) is a solution of D.E. (ii).

5. $y = Ax : xy' = y$ $(x \neq 0)$

Sol. Given: $y = Ax$ ...(i)

To prove: y given by (i) is a solution of the D.E. $xy' = y$ $(x \neq 0)$ ...(ii)

From (i), $y' = A(1) = A$

L.H.S. of D.E. (ii) $= xy' = xA$

$= Ax = y$ [By (i)] = R.H.S. of D.E. (ii)

$\therefore$ y given by (i) is a solution of D.E. (ii).

6. $y = x \sin x : xy' = y + x\sqrt{x^2 - y^2}$ **(**$x \neq 0$ **and** $x > y$ **or** $x < -y$**)**

Sol. Given: $y = x \sin x$ ...(i)

To prove: y given by (i) is a solution of D.E.

$xy' = y + x\sqrt{z^2 - y^2}$ ...(ii) $(x \neq 0$ and $x > y$ or $x < -y)$

From (i), $\dfrac{ay}{az} (= y') = x \dfrac{a}{az} (\sin x) + \sin x \dfrac{a}{az} x = x \cos x + \sin x$

L.H.S. of D.E. (ii) $= xy' = x(x \cos x + \sin x)$

$= x^2 \cos x + x \sin x$ ...(iii)

R.H.S. of D.E. (ii) $= y + x\sqrt{z^2 - y^2}$

Putting $y = x \sin x$ from (i),

$= x \sin x + x \sqrt{z^2 - z^2 \sin^2 z} = x \sin x + x \sqrt{z^2 (1 - \sin^2 z)}$

$= x \sin x + x \sqrt{z^2 \cos^2 z} = x \sin x + x \cdot x \cos x$

$= x \sin x + x^2 \cos x = x^2 \cos x + x \sin x$ ...(*iv*)

From (*iii*) and (*iv*), L.H.S. of D.E. (*ii*) = R.H.S. of D.E. (*ii*)

$\therefore$ y given by (*i*) is a solution of D.E. (*ii*).

In each of the Exercises 7 to 10, verify that the given functions (Explicit or Implicit) is a solution of the corresponding differential equation:

7. $xy = \log y + C : y' = \dfrac{y^2}{1 - xy}$ $(xy \neq 1)$

Sol. Given: $xy = \log y + C$ ...(*i*)

To prove that Implicit function given by (*i*) is a solution of the

D.E. $y' = \dfrac{y^2}{1 - zy}$ ...(*ii*)

Differentiating both sides of (*i*) w.r.t. *x*, we have

$$xy' + y(1) = \frac{1}{y} y' + 0$$

$$\Rightarrow \quad xy' - \frac{y'}{y} = -y \qquad \Rightarrow \quad y'\left(z - \frac{1}{y}\right) = -y$$

$$\Rightarrow \quad y'\left(\frac{zy - 1}{y}\right) = -y \qquad \Rightarrow \quad y'(xy - 1) = -y^2$$

$$\Rightarrow \quad y' = \frac{-y^2}{zy - 1} = \frac{-\ \ y^2}{-(1 - zy)} \quad \frac{y^2}{1 - zy}$$

which is same as differential equation (*ii*), *i.e.,* Eqn. (*ii*) is proved.

$\therefore$ Function (Implicit) given by (*i*) is a solution of D.E. (*ii*).

8. $y - \cos y = x : (y \sin y + \cos y + x) y' = y$

Sol. Given: $y - \cos y = x$ ...(*i*)

To prove that function given by (*i*) is a solution of D.E.

$(y \sin y + \cos y + x) y' = y$ ...(*ii*)

Differentiating both sides of (*i*) w.r.t. *x*, we have

$$y' + (\sin y) y' = 1 \qquad \Rightarrow \quad y'(1 + \sin y) = 1$$

$\Rightarrow$ $y' = \dfrac{1}{1 + \sin y}$ ...(*iii*)

Putting the value of *x* from (*i*) and value of *y'* from (*iii*) in L.H.S. of (*ii*), we have

L.H.S. $= (y \sin y + \cos y + x) y'$

$$= (y \sin y + \cos y + y - \cos y) \frac{1}{1 + \sin y} = (y \sin y + y) \frac{1}{1 + \sin y}$$

$$= y(\sin y + 1) \frac{1}{(1 + \sin y)} = y = \text{R.H.S. of } (ii).$$

$\therefore$ The function given by (*i*) is a solution of D.E. (*ii*).

9. $x + y = \tan^{-1} y : y^2 y' + y^2 + 1 = 0$

Sol. Given: $x + y = \tan^{-1} y$ ...(*i*)

To prove that function given by (*i*) is a solution of D.E.

$$y^2 y' + y^2 + 1 = 0 \quad \text{...(ii)}$$

Differentiating both sides of (*i*), w.r.t. x, $1 + y' = \frac{1}{1 + y^2} y'$

Cross-multiplying

$(1 + y')(1 + y^2) = y' \quad \Rightarrow \quad 1 + y^2 + y' + y'y^2 = y'$

$\Rightarrow$ $y^2y' + y^2 + 1 = 0$ which is same as D.E. (*ii*).

$\therefore$ Function given by (*i*) is a solution of D.E. (*ii*).

10. $y = \sqrt{a^2 - x^2}$, $x \in (-a, a) : x + y \frac{dy}{dx} = 0 \ (y \neq 0)$

Sol. Given: $y = \sqrt{a^2 - z^2}$, $x \in (-a, a)$ ...(*i*)

To prove that function given by (*i*) is a solution of D.E.

$$x + y \frac{ay}{az} = 0 \quad ...(ii)$$

From (*i*), $\frac{ay}{az} = \frac{1}{2} (a^2 - x^2)^{-1/2} \frac{a}{az} (a^2 - x^2)$

$$= \frac{1}{2\sqrt{a^2 - z^2}} (-2x) = \frac{-z}{\sqrt{a^2 - z^2}} \quad ...(iii)$$

Putting these values of y and $\frac{ay}{az}$ from (*i*) and (*iii*) in L.H.S. of (*ii*),

$$\text{L.H.S.} = x + y \frac{ay}{az} = x + \sqrt{a^2 - z^2} \left(\frac{-z}{\sqrt{a^2 - z^2}} \right)$$

$= x - x = 0 =$ R.H.S. of D.E. (*ii*).

$\therefore$ Function given by (*i*) is a solution of D.E. (*ii*).

11. Choose the correct answer:

The number of arbitrary constants in the general solution of a differential equation of fourth order are:

(A) 0 **(B) 2** **(C) 3** **(D) 4.**

Sol. Option (D) 4 is the correct answer.

Result. The number of arbitrary constants (c_1, c_2, c_3 etc.) in the general solution of a differential equation of nth order is n.

12. The number of arbitrary constants in the particular solution of a differential equation of third order are

(A) 3 **(B) 2** **(C) 1** **(D) 0.**

Sol. The number of arbitrary constants in a particular solution of a differential equation of any order is zero (0).

[$\because$ By definition, a particular solution is a solution which contains no arbitrary constant.]

$\therefore$ Option (D) is the correct answer.

Exercise 9.3

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants *a* and *b*.

1. $\frac{x}{a} + \frac{y}{b} = 1$

Sol. Equation of the given family of curves is $\frac{z}{a} + \frac{y}{b} = 1$ ...(*i*)

Here there are two arbitrary constants *a* and *b*. So we shall differentiate both sides of (*i*) two times w.r.t. *x*.

From (*i*), $\frac{1}{a} \cdot 1 + \frac{1}{b}\frac{ay}{az} = 0$ or $\frac{1}{a} = -\frac{1}{b}\frac{ay}{az}$ ...(*ii*)

Again diff. (*ii*) w.r.t. *x*, $0 = -\frac{1}{b}\frac{a^2y}{az^2}$

Multiplying both sides by – *b*, $\frac{a^2y}{az^2} = 0$.

Which is the required D.E.

Remark. We need not eliminate *a* and *b* because they have already got eliminated in the process of differentiation.

2. $y^2 = a(b^2 - x^2)$

Sol. Equation of the given family of curves is

$$y^2 = a(b^2 - x^2) \quad ...(i)$$

Here there are two arbitrary constants *a* and *b*. So, we are to differentiate (*i*) twice w.r.t. *x*.

From (*i*), $2y\frac{ay}{az} = a(0 - 2x) = -2ax$.

Dividing by 2, $y\frac{ay}{az} = -ax$ ...(*ii*)

Again differentiating both sides of (*ii*) w.r.t. *x*,

$y\frac{a^2y}{az^2} + \frac{ay}{az} \cdot \frac{ay}{az} = -a$ or $y\frac{a^2y}{az^2} + \left(\frac{ay}{az}\right)^2 = -a$ ...(*iii*)

Putting this value of – *a* from (*iii*) in (*ii*), (To eliminate *a*, as *b* is already absent in both (*ii*) and (*iii*)), we have

$y\frac{ay}{az} = \left(y\frac{a^2y}{az^2} + \left(\frac{ay}{az}\right)^2\right)x$ or $xy\frac{a^2y}{az^2} + x\left(\frac{ay}{az}\right)^2 = y\frac{ay}{az}$

or $xy\frac{a^2y}{az^2} + x\left(\frac{ay}{az}\right)^2 - y\frac{ay}{az} = 0.$

3. $y = ae^{3x} + be^{-2x}$

Sol. Equation of the family of curves is

$$y = a\,e^{3x} + b\,e^{-2x} \quad ...(i)$$

Here are two arbitrary constants *a* and *b*.

From (*i*), $\frac{ay}{az} = 3\,ae^{3x} - 2\,be^{-2x}$ ...(*ii*)

Again differentiating both sides of (*ii*), w.r.t. *x*,

$$\frac{a^2y}{az^2} = 9\,ae^{3x} + 4\,be^{-2x} \qquad \ldots(iii)$$

Let us eliminate *a* and *b* from (*i*), (*ii*) and (*iii*).

Equation (*ii*) – 3 × eqn. (*i*) gives (To eliminate *a*),

$$\frac{dy}{dx} - 3y = -5\, be^{-2x} \quad ...(iv)$$

Again Eqn. (*iii*) – 3 × eqn (*ii*) gives (again to eliminate *a*)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 10\, be^{-2x} \quad ...(v)$$

Now Eqn. (*v*) + 2 × eqn. (*iv*) gives (To eliminate *b*)

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2\left(\frac{dy}{dx} - 3y\right) = 10\, be^{-2x} - 10\, be^{-2x}$$

or $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2\frac{dy}{dx} - 6y = 0$$

or $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

which is the required D.E.

4. $\boldsymbol{y = e^{2x}(a + bx)}$

Sol. Equation of the given family of curves is

$$y = e^{2x}(a + bx) \quad ...(i)$$

Here are two arbitrary constants *a* and *b*.

From (*i*), $$\frac{dy}{dx} = \left(\frac{d}{dx} e^{2x}\right)(a + bx) + e^{2x}\frac{d}{dx}(a + bx)$$

or $$\frac{dy}{dx} = 2\, e^{2x}(a + bx) + e^{2x} \,.\, b$$

or $$\frac{dy}{dx} = 2y + be^{2x} \quad ...(ii)$$

(By (*i*))

Again differentiating both sides of (*ii*), w.r.t. *x*

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 2\, be^{2x} \quad ...(iii)$$

Let us eliminate *b* from eqns. (*ii*) and (*iii*), (as *a* is already absent in both (*ii*) and (*iii*))

From eqn. (*ii*) $\frac{dy}{dx} - 2y = be^{2x}$

Putting this value of be^{2x} in (*iii*), we have

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 2\left(\frac{dy}{dx} - 2y\right) \Rightarrow \frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 2\frac{dy}{dx} - 4y$$

or $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

which is the required D.E.

5. $\boldsymbol{y = e^x(a \cos x + b \sin x)}$

Sol. Equation of family of curves is

$$y = e^x (a \cos x + b \sin x) \qquad \ldots(i)$$

$$\therefore \quad \frac{ay}{az} = \left(\frac{a}{az} e^z\right) (a \cos x + b \sin x) + e^x (- a \sin x + b \cos x)$$

or $\frac{ay}{az} = e^x (a \cos x + b \sin x) + e^x (-a \sin x + b \cos x)$

or $\frac{ay}{az} = y + e^x (-a \sin x + b \cos x)$ ...(*ii*)

(By (*i*))

Again differentiating both sides of eqn. (*ii*), w.r.t. *x*, we have

$\frac{a^2y}{az^2} = \frac{ay}{az} + e^x (-a \sin x + b \cos x) + e^x (-a \cos x - b \sin x)$

or $\frac{a^2y}{az^2} = \frac{ay}{az} + \left(\frac{ay}{az} - y\right) - e^x (a \cos x + b \sin x)$

(By (*ii*))

or $\frac{a^2y}{az^2} = 2\frac{ay}{az} - y - y$

(By (*i*))

or $\frac{a^2y}{az^2} \quad 2\frac{ay}{az} + 2y = 0$ which is the required D.E.

6. Form the differential equation of the family of circles touching the *y*-axis at the origin.

Sol. Clearly, a circle which touches *y*-axis at the origin must have its centre on *x*-axis.

[∵ *x*-axis being at right angles to tangent *y*-axis is the normal or line of radius of the circle.]

∴ **The centre of circle is (*r*, 0) where *r* is the radius of the circle.**

∴ Equation of required circles is

$(x - r)^2 + (y - 0)^2 = r^2$ $[(x - \alpha)^2 + (y - \beta)^2 = r^2]$

or $x^2 + r^2 - 2rx + y^2 = r^2$

or $x^2 + y^2 = 2rx$ ...(*i*)

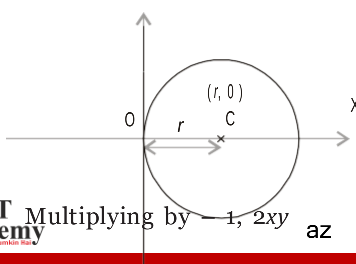
where *r* is the only arbitrary constant.

∴ Differentiating both sides of (*i*) only once w.r.t. *x*, we have

$2x + 2y\frac{ay}{az} = 2r$ ...(*ii*)

To eliminate *r*, putting the value of 2*r* from (*ii*) in (*i*),

$x^2 + y^2 = \left(2z + 2y\frac{ay}{az}\right)x$



or $x^2 + y^2 = 2x^2 + 2xy\frac{ay}{az}$

or $-2xy\frac{ay}{az} - x^2 + y^2 = 0$

Multiplying by -1, $2xy\frac{ay}{az}$

$+ x^2 - y^2 = 0$

or $2xy \frac{ay}{az} + x^2 = y^2$ which is the required D.E.

Remark. The above question can also be stated as : **Form the D.E. of the family of circles passing through the origin and having centres on *x*-axis.**

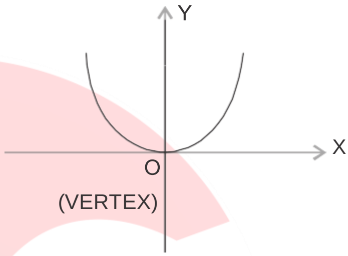
7. Find the differential equation of the family of parabolas having vertex at origin and axis along positive *y*-axis.

Sol. We know that equation of parabolas having vertex at origin and axis along positive *y*-axis is $x^2 = 4ay$ ...(*i*)

Here *a* is the only arbitrary constant. So differentiating both sides of Eqn. (*i*) only once w.r.t. *x*, we have

$$2x = 4a \frac{ay}{az} \quad ...(ii)$$

To eliminate *a*, putting



$4a = \frac{z^2}{y}$ from (*i*) in (*ii*), we have

$$2x = \frac{z^2}{y} \frac{ay}{az}$$

$$\Rightarrow \quad 2xy = x^2 \frac{ay}{az}$$

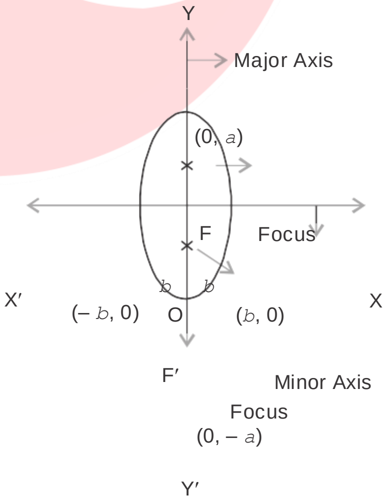
Dividing both sides by *x*, $2y = x \frac{ay}{az}$

$$\Rightarrow \quad x\frac{ay}{az} + 2y = 0$$

$$\Rightarrow \quad x\frac{ay}{az} - 2y = 0$$ which is the required D.E.

8. Form the differential equation of family of ellipses having foci on *y*-axis and centre at the origin.

Sol. We know that equation of ellipses having foci on *y*-axis *i.e.,* vertical ellipses with major axis as *y*-axis is



$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \quad ...(i)$$

Here *a* and *b* are two arbitrary constants.

So we shall differentiate eqn. (i) twice w.r.t. *x*.

Differentiating both sides of (i) w.r.t. *x*, we have

1 ay 1 a 2 2y az

$+ \; b^2 \qquad 2x = 0$

or $\quad \frac{2}{a^2} y \frac{ay}{az} = - \frac{2}{b^2} x$

Dividing both sides by 2,

$$\frac{1}{a^2} y \frac{ay}{az} = \frac{-1}{b^2} x \qquad ...(ii)$$

Again differentiating both sides of (*ii*) w.r.t. *x*, we have

$$\frac{1}{a^2}\left[y \frac{a^2y}{az^2} + \frac{ay}{az} . \frac{ay}{az}\right] = \frac{-1}{b^2} \qquad ...(iii)$$

To eliminate *a* and *b*, putting this value of $\frac{-1}{b^2}$ from (*iii*) in (*ii*), the required differential equation is

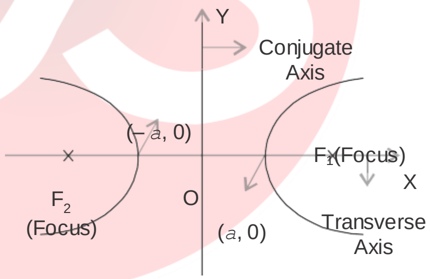
$$\frac{1}{a^2} y \frac{ay}{az} = \frac{1}{a^2}\left[y \frac{a^2y}{az^2} + \left(\frac{ay}{az}\right)^2\right] x$$

Multiplying both sides by a^2, $y \frac{ay}{az} = xy \frac{a^2y}{az^2} + x\left(\frac{ay}{az}\right)^2$

or $xy \frac{a^2 y}{az^2} + x\left(\frac{ay}{az}\right)^2 - y\frac{ay}{az} = 0$

which is the required differential equation.

9. Form the differential equation of the family of hyperbolas having foci on *x*-axis and centre at the origin.



Sol. We know that equation of hyperbolas having foci on *x*-axis and centre at origin is

$$\frac{z^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Here *a* and *b* are two arbitrary constants. So we shall differentiate eqn. (*i*) twice w.r.t. *x*.

From (*i*), $\frac{1}{a^2} . 2x - \frac{1}{b^2} . 2y \frac{ay}{az} = 0$ or $\frac{2}{a^2} x = \frac{2}{b^2} y \frac{ay}{az}$

Dividing both sides by 2, $\frac{1}{a^2} x = \frac{1}{b^2} y \frac{ay}{az} \qquad ...(ii)$

Again differentiating both sides of (*ii*), w.r.t. *x*,

$$\frac{1}{a^2}\cdot 1 = \frac{1}{b^2}\cdot\left[y\frac{a^2y}{az^2} + \frac{ay}{az}\cdot\frac{ay}{az}\right]$$

or $$\frac{1}{a^2} = \frac{1}{b^2}\left[y\frac{a^2y}{az^2} + \left(\frac{ay}{az}\right)^2\right] \qquad ...(iii)$$

Dividing eqn. (*iii*) by eqn. (*ii*), we have (To eliminate *a* and *b*)

$$\frac{1}{z} = \frac{y\frac{a^2y}{az^2} + \left(\frac{ay}{az}\right)^2}{y\frac{ay}{az}}$$

Cross-multiplying, $x\left(y\frac{a^2y}{az^2} + \left(\frac{ay}{az}\right)^2\right) = y\frac{ay}{az}$

or $xy\frac{a^2y}{az^2} + x\left(\frac{ay}{az}\right)^2 - y\frac{ay}{az} = 0$

which is the required differential equation.

10. Form the differential equation of the family of circles having centres on *y*-axis and radius 3 units.

Sol. We know that on y-axis, $x = 0$.

∴ Centre of the circle on y-axis is $(0, \beta)$.

∴ Equation of the circle having centre on y-axis and radius 3 units is

$$(x - 0)^2 + (y - \beta)^2 = 3^2 \qquad [(x - \alpha)^2 + (y - \beta)^2 = r^2]$$

or $\qquad x^2 + (y - \beta)^2 = 9 \qquad \ldots(i)$

Here β is the only arbitrary constant. So we shall differentiate both sides of eqn. (i) only once w.r.t. x,

From (i), $2x + 2(y - \beta)\frac{a}{az}(y - \beta) = 0$

or $2x + 2(y - \beta)\frac{ay}{az} = 0$

or $2(y - \beta)\frac{ay}{az} = -2x \quad ∴ \quad y - \beta = \frac{-2z}{2\frac{ay}{az}} = \frac{-z}{\frac{ay}{az}}$

Putting this value of $(y - \beta)$ in (i) (To eliminate β), we have

$$x^2 + \frac{z^2}{\left(\frac{ay}{az}\right)^2} = 9$$

L.C.M. $= \left(\frac{ay}{az}\right)^2$. Multiplying both sides by this L.C.M.,

$$x^2\left(\frac{ay}{az}\right)^2 + x^2 = 9\left(\frac{ay}{az}\right)^2$$

$\Rightarrow x^2\left(\frac{ay}{az}\right)^2 - 9\left(\frac{ay}{az}\right)^2 + x^2 = 0$ or $(x^2 - 9)\left(\frac{ay}{az}\right)^2 + x^2 = 0$

which is the required differential equation.

11. Which of the following differential equation has $y = c_1 e^x$

+ $c_2 e^{-x}$ as the general solution?

(A) $\frac{d^2y}{dx^2} + y = 0$ (B) $\frac{d^2y}{dx^2} - y = 0$

(C) $\frac{d^2y}{dx^2} + 1 = 0$ **(D)** $\frac{d^2y}{dx^2} - 1 = 0$

Sol. Given: $y = c_1 e^x + c_2 e^{-x}$ ...(*i*)

$\therefore \frac{ay}{az} = c_1 e^x + c_2 e^{-x}(-1) = c_1 e^x - c_2 e^{-x}$

$\therefore \frac{a^2y}{az^2} = c_1 e^x - c_2 e^{-x}(-1) = c_1 e^x + c_2 e^{-x}$

or $\frac{a^2y}{az^2} = y$ [By (*i*)]

or $\frac{a^2y}{az^2} - y = 0$ which is differential equation given in option (B)

$\therefore$ Option (B) is the correct answer.

12. Which of the following differential equations has $y = x$ as one of its particular solutions?

(A) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = x$ **(B)** $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = x$

(C) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$ **(D)** $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = 0$

Sol. Given: $y = x$

$\therefore \frac{ay}{az} = 1$ and $\frac{a^2y}{az^2} = 0$

These values of y, $\frac{ay}{az}$ and $\frac{a^2y}{az^2}$ clearly satisfy the D.E. of option (C).

[$\because$ L.H.S. of D.E. of option (C) $= \frac{a^2y}{az^2} - x^2 \frac{ay}{az} + xy$

$= 0 - x^2(1) + x(x) = -x^2 + x^2 = 0 =$ R.H.S. of option (C)]

$\therefore$ Option (C) is the correct answer.

Exercise 9.4 *(Page No. 395-397)*

For each of the differential equations in Exercises 1 to 4, find the general solution:

1. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

Sol. The given differential equation is or $dy =$ $2\sin^2 \frac{z}{}$

$\frac{ay}{az} = \frac{1-\cos z}{1+\cos z}$ $\frac{1-\cos z}{1+\cos z}$

dx.

Integrating both sides, $\int ay = \frac{2}{2\cos^2\frac{z}{2}}\,dx$

or $y = \int \tan^2\frac{z}{2}\,dx = \int \left(\sec^2\frac{z}{2} - 1\right)dx = \frac{\tan\frac{z}{2}}{\frac{1}{2}} - x + c$

Exercise 9.4

For each of the differential equations in Exercises 1 to 4, find the general solution:

1. $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$

Sol. The given differential equation is

$$\frac{ay}{az} = \frac{1 - \cos z}{1 + \cos z} \quad \text{or} \quad dy = \frac{1 - \cos z}{1 + \cos z}\, dx.$$

Integrating both sides, $\int ay = \int \frac{2 \sin^2 \frac{z}{2}}{2 \cos^2 \frac{z}{2}}\, dx$

or $y = \int \tan^2 \frac{z}{2}\, dx = \int \left(\sec^2 \frac{z}{2} - 1 \right) dx = \frac{\tan \frac{z}{2}}{\frac{1}{2}} - x + c$

or $y = 2 \tan \frac{z}{2} - x + c$

which is the required general solution.

2. $\frac{dy}{dx} = \sqrt{4 - y^2}$ **$(-2 < y < 2)$**

Sol. The given D.E. is $\frac{ay}{az} = \sqrt{4 - y^2} \Rightarrow dy = \sqrt{4 - y^2}\, dx$

Separating variables, $\frac{ay}{\sqrt{4 - y^2}} = dx$

Integrating both sides, $\int \frac{ay}{\sqrt{2^2 - y^2}} dy = \int 1\, dx$

$\therefore \sin^{-1} \frac{y}{2} = x + c$ $\left[\because \int \frac{1}{\sqrt{a^2 - z^2}} az = \sin^{-1} \frac{z}{a}\right]$

$\Rightarrow \frac{y}{2} = \sin (x + c)$

$\Rightarrow y = 2 \sin (x + c)$ which is the required general solution.

3. $\frac{dy}{dx}$ **$+ y = 1 \quad (y \neq 1)$**

Sol. The given differential equation is $\frac{ay}{az} + y = 1$

$\Rightarrow \frac{ay}{az} = 1 - y \Rightarrow dy = (1 - y)\, dx \Rightarrow dy = -(y - 1)\, dx$

Separating variables, $\frac{ay}{y - 1} = -dx$

Integrating both sides, $\int \frac{ay}{y - 1} = -\int 1\, dx$

$\Rightarrow \log |y - 1| = -x + c$

$\Rightarrow |y - 1| = e^{-x + c}$ $[\because$ If $\log x = t$, then $x = e^t]$

$\Rightarrow y - 1 = \pm e^{-x + c} \Rightarrow y = 1 \pm e^{-x} e^c$

$\Rightarrow y = 1 \pm e^c e^{-x}$

$\Rightarrow y = 1 + Ae^{-x}$ where $A = \pm e^c$

which is the required general solution.

4. **$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$**

Sol. The given differential equation is

$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Dividing by tan x tan y, we have

$$\frac{\sec^2 z}{\tan z}\,dx + \frac{\sec^2 y}{\tan\,y}\,dy = 0 \qquad \text{(Variables separated)}$$

Integrating both sides, $\int \frac{\sec^2 z}{\tan z}\,dx + \int \frac{\sec^2 y}{\tan\,y}\,dy = \log c$

or $\log|\tan x| + \log|\tan y| = \log c$ $\left[\because \int \frac{f'(z)}{f(z)} az = \log\|f(z)\|\right]$

or $\log|(\tan x \tan y)| = \log c$ or $|\tan x \tan y| = c$

$\therefore$ $\tan x \tan y = \pm c = C$ where $C = \pm c$.

$[\because |t| = a(a \geq 0) \Rightarrow t = \pm a]$

which is the required general solution.

For each of the differential equations in Exercises 5 to 7, find the general solution:

5. $(e^x + e^{-x})\, dy - (e^x - e^{-x})\, dx = 0$

Sol. The given D.E. is $(e^x + e^{-x})\, dy = (e^x - e^{-x})\, dx$

or $dy = \left(\frac{e^z - e^{-z}}{e^z + e^{-z}}\right) dx$

Integrating both sides, $\int ay = \int \left(\frac{e^z - e^{-z}}{e^z + e^{-z}}\right) dx$

or $y = \log|e^x + e^x| + c$ $\left[\because \int \frac{f'(z)}{f(z)} az = \log\|f(z)\|\right]$

which is the required general solution.

6. $\frac{dy}{dx}$ **= (1 + x^2)(1 + y^2)**

Sol. The given differential equation is $\frac{ay}{az} = (1 + x^2)(1 + y^2)$

$\Rightarrow$ $dy = (1 + x^2)(1 + y^2)\, dx$

Separating variables, $\frac{ay}{1 + y^2} = (1 + x^2)\, dx$

Integrating both sides,

$\int \frac{1}{y^2 + 1}\, dy = \int (z^2 + 1)\, dx \Rightarrow \tan^{-1} y = \frac{z^3}{3} + x + c$

which is the required general solution.

7. $y \log y\, dx - x\, dy = 0$

Sol. The given differential equation is $y \log y\, dx - x\, dy = 0$

$\Rightarrow$ $-x\, dy = -y \log y\, dx$

Separating variables, $\frac{ay}{y \log y} = \frac{az}{z}$ ...(*i*)

Integrating both sides $\int \frac{ay}{y \log y} = \int \frac{ay}{z}$

For integral on left hand side, put $\log y = t$.

$$\therefore \quad \frac{1}{y} = \frac{at}{ay} \qquad \Rightarrow \quad \frac{ay}{at^{y}} = dt$$

az

$$\therefore \quad \text{Eqn. } (i) \text{ becomes } \int \frac{}{t} = \int \frac{}{z}$$

$$\Rightarrow \quad \log |t| = \log |x| + \log |c|^{*} \qquad \ldots(ii)$$

$$= \log |xc|$$

$\Rightarrow \quad | t | = | xc |$

$\Rightarrow \quad t = \pm xc$

$[\because \ | x | = | y | \ \Rightarrow \ x = \pm y]$

$\Rightarrow \quad \log y = \pm xc = ax \quad \text{where } a = \pm c$

$\therefore \ y = e^{ax}$ which is the required general solution.

For each of the differential equations in Exercises 8 to 10, find the general solution:

8. $x^5 \dfrac{dy}{dx} = -y^5$

Sol. The given differential equation is $x^5 \dfrac{ay}{az} = -y^5$

$\Rightarrow \quad x^5\, dy = -y^5\, dx$

Separating variables, $\dfrac{ay}{(y^5)} = -\dfrac{az}{(z^5)} \quad \Rightarrow \quad y^{-5}\, dy = -x^{-5}\, dx$

Integrating both sides, $\int y^{-5}\, dy = -\int z^{-5}\, dx$

$$\frac{y^{-4}}{-4} = -\frac{z^{-4}}{-4} + c$$

Multiplying by – 4, $\quad y^{-4} = -x^{-4} - 4c$

$\Rightarrow \quad x^{-4} + y^{-4} = -4c \quad \Rightarrow \quad x^{-4} + y^{-4} = C$ where $C = -4c$

which is the required general solution.

9. $\dfrac{dy}{dx} = \sin^{-1} x$

Sol. The given differential equation is $\dfrac{ay}{az} = \sin^{-1} x$

or $\quad dy = \sin^{-1} x\, dx$

Integrating both sides, $\quad \int 1\, dy = \int \sin^{-1} z\, dx$

or $\quad y = \int \underset{\text{I}}{\sin^{-1} z} \,.\, \underset{\text{II}}{1}\, dx$

Applying product rule,

$$y = (\sin^{-1} x) \int 1\, dx - \int \frac{a}{az} (\sin^{-1} x) \int 1\, dx\, dx$$

$$= x \sin^{-1} x - \int \frac{1}{}\, x\, dx \qquad \ldots(i)$$

$$\sqrt{1-z^2}$$

To evaluate $\int \frac{z}{\sqrt{1-z^2}}\, dx = -\frac{1}{2}\int \frac{-2z}{\sqrt{1-z^2}}\, dx$

Put $1 - x^2 = t$. Differentiate $-2x\, dx = dt$

***Remark. To explain * in eqn. (*ii*)**

If all the terms in the solution of a D.E. involve logs, it is better to use log c or log $|c|$ instead of c in the solution.

$$\therefore \int \frac{z}{\sqrt{1-z^2}}\, dx = -\frac{1}{2}\int \frac{at}{\sqrt{t}} = -\frac{1}{2}\int t^{-1/2}\, dt$$

$$= -\frac{1}{2}\frac{t^{1/2}}{1/2} = -\sqrt{t} = -\sqrt{1-z^2}$$

Putting this value of $\int \frac{z}{\sqrt{1-z^2}}\, dx$ in (i), the required general solution is

$$y = x\sin^{-1} x + \sqrt{1-z^2} + c.$$

10. $\boldsymbol{e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0}$

Sol. The given equation is $e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$

Dividing every term by $(1 - e^x)\tan y$, we have

$$\frac{e^z}{1-e^z}\, dx + \frac{\sec^2 y}{\tan y}\, dy = 0 \quad \text{(Variables separated)}$$

Integrating both sides, $\int \frac{e^z}{1-e^z}\, dx + \int \frac{\sec^2 y}{\tan y}\, dy = c$

or $\quad -\int \frac{-e^z}{1-e^z}\, dx + \log|\tan y| = c$

or $\quad -\log|1-e^x| + \log|\tan y| = c \quad \left[\because \int \frac{f'(z)}{f(z)}\, az = \log|f(z)|\right]$

or $\quad \log\frac{|\tan y|}{|1-e^z|} = \log c' \quad$ (See Remark at the end of page 612)

or $\quad \frac{|\tan y|}{|1-e^z|} = c'$

or $\quad \tan y = C(1 - e^x)$. $\quad [\because |t| = c' \Rightarrow t = \pm c' = C \text{ (say)}]$

For each of the differential equations in Exercises 11 to 12, find a particular solution satisfying the given condition:

11. $\boldsymbol{(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x,\ y = 1, \text{ when } x = 0}$

Sol. The given differential equation is $(x^3 + x^2 + x + 1)\frac{ay}{az} = 2x^2 + x$

$\therefore \quad (x^3 + x^2 + x + 1)\, dy = (2x^2 + x)\, dx$

Separating variables $dy = \dfrac{(2z^2 + z)}{z^3 + z^2 + z + 1}\, dx$

or $dy = \dfrac{2z^2 + z}{(z+1)(z^2+1)}\, dx$

$[\because \; x^3 + x^2 + x + 1 = x^2(x + 1) + (x + 1) = (x + 1)(x^2 + 1)]$

Integrating both sides, we have

$$\int 1 \; dy = \int \frac{2z^2 + z}{(z+1)(z^2+1)} \, dx \quad \text{or} \quad y = \int \frac{2z^2 + z}{(z+1)(z^2+1)} \, dx \quad ...(i)$$

Let $\dfrac{2z^2 + z}{(z+1)(z^2+1)} = \dfrac{A}{z+1} + \dfrac{Bz + C}{z^2+1}$ (Partial fractions)

...(*ii*)

Multiplying both sides by L.C.M. = $(x + 1)(x^2 + 1)$, we have

$2x^2 + x = A(x^2 + 1) + (Bx + C)(x + 1)$

or $2x^2 + x = Ax^2 + A + Bx^2 + Bx + Cx + C$

Comparing coeff. of x^2 on both sides, we have

$$A + B = 2 \quad ...(iii)$$

Comparing coeff. of x on both sides, we have

$$B + C = 1 \quad ...(iv)$$

Comparing constants $A + C = 0$ ...(*v*)

Let us solve eqns. (*iii*), (*iv*) and (*v*) for A, B, C

eqn. (*iii*) – eqn. (*iv*) gives to eliminate B,

$$A - C = 1 \quad ...(vi)$$

Adding (*v*) and (*vi*), $2A = 1$ or $A = \dfrac{1}{2}$

From (*v*), $C = -A = -\dfrac{1}{2}$

Putting $C = -\dfrac{1}{2}$ in (*iv*), $B - \dfrac{1}{2} = 1$ or $B = 1 + \dfrac{1}{2} = \dfrac{3}{2}$

Putting these values of A, B, C in (*ii*), we have

$$\frac{2z^2 + z}{(z+1)(z^2+1)} = \frac{\frac{1}{2}}{z+1} + \frac{\frac{3}{2}z - \frac{1}{2}}{z^2+1}$$

$$= \frac{1}{2}\frac{1}{z+1} + \frac{3}{2} \cdot \frac{z}{z^2+1} - \frac{1}{2}\frac{1}{z^2+1}$$

$$= \frac{1}{2}\frac{1}{z+1} + \frac{3}{4} \cdot \frac{2z}{z^2+1} - \frac{1}{2}\frac{1}{z^2+1}$$

Putting this value in (*i*)

$$y = \frac{1}{2}\int \frac{1}{z+1} \, dx + \frac{3}{4}\int \frac{2z}{z^2+1} \, dx - \frac{1}{2}\int \frac{1}{z^2+1} \, dx$$

$$y = \frac{1}{2}\log(x + 1) + \frac{3}{4}\log(x^2 + 1) - \frac{1}{2}\tan^{-1} x + c \quad ...(vii)$$

$$\left[\int \frac{2z}{z^2} \, az = \int \frac{f'(z)}{f\;z} \, az = \log f(z)\right]$$

$\lfloor$ z +1 ( ) $\rfloor$

To find *c*

When $x = 0$, $y = 1$ (given)

Putting $x = 0$ and $y = 1$ in (*vii*),

$$1 = \frac{1}{2}\log 1 + \frac{3}{4}\log 1 - \frac{1}{2}\tan^{-1} 0 + c$$

or $1 = c$ [$\because \log 1 = 0$ and $\tan^{-1} 0 = 0$]

Putting $c = 1$ in eqn. (*vii*), the required solution is

$$y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + 1.$$

$$y = \frac{1}{4}[2\log(x+1) + 3\log(x^2+1)] - \frac{1}{2}\tan^{-1}x + 1$$

$$= \frac{1}{4}[\log(x+1)^2 + \log(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + 1$$

$$= \frac{1}{4}[\log(x+1)^2(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + 1$$

which is the required particular solution.

12. $x(x^2 - 1)\frac{dy}{dx} = 1$; $y = 0$ **when** $x = 2$.

Sol. The given differential equation is $x(x^2 - 1)\frac{ay}{az} = 1$

$\Rightarrow \quad x(x^2 - 1)\,dy = dx \qquad \Rightarrow \quad dy = \frac{az}{z(z^2 - 1)}$

Integrating both sides, $\int 1\,dy = \int \frac{1}{z(z^2-1)}\,dx$

$\Rightarrow \quad y = \int \frac{1}{z(z+1)(z-1)}\,dx + c$ ...(*i*)

Let the integrand $\frac{1}{z(z+1)(z-1)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-1}$ ...(*ii*)

(By Partial Fractions)

Multiplying by L.C.M. $= x(x+1)(x-1)$,

$1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$

or $1 = A(x^2 - 1) + B(x^2 - x) + C(x^2 + x)$

or $1 = Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2**:** $A + B + C = 0$ ...(*iii*)

x**:** $-B + C = 0 \quad \Rightarrow \quad C = B$ ...(*iv*)

Constants $-A = 1$ or $A = -1$

Putting $A = -1$ and $C = B$ from (*iv*) in (*iii*),

$-1 + B + B = 0$ or $2B = 1 \qquad \Rightarrow \quad B = \frac{1}{2}$

$\therefore$ From (*iv*), C = B = $\frac{1}{2}$

Putting these values of A, B, C in (*ii*),

$$\frac{1}{z(z+1)(z-1)} = \frac{-1}{z} + \frac{\frac{1}{2}}{z+1} + \frac{\frac{1}{2}}{z-1}$$

$$\therefore \int \frac{+\ 1 \quad -}{z(z \quad 1)(z \quad 1)} dx = -\int \frac{1}{z} dx + \frac{1}{2}\int \frac{1}{z+1} dx + \frac{1}{2}\int \frac{1}{z-1} dx$$

$$= -\log|x| + \frac{1}{2}\log|x+1| + \frac{1}{2}\log|x-1|$$

$$= \frac{1}{2}[-2\log|x| + \log|x+1| + \log|x-1|]$$

$$= \frac{1}{2}[-\log|x|^2 + \log|(x+1)(x-1)|]$$

$$\Rightarrow \int \frac{1}{z(z+1)(z-1)} dx = \frac{1}{2}\left[\log\frac{|z^2-1|}{|z|^2}\right] = \frac{1}{2}\log\frac{|z^2-1|}{z^2}$$

Putting this value in (*i*),

$$y = \frac{1}{2}\log\left|\frac{z^2-1}{z^2}\right| + c \qquad ...(v)$$

To find *c* for the particular solution

Putting $y = 0$, when $x = 2$ (given) in (*v*),

$$0 = \frac{1}{2}\log\frac{3}{4} + c \qquad \Rightarrow c = \frac{-1}{2}\log\frac{3}{4}$$

Putting this value of *c* in (*v*), the required particular solution is

$$y = \frac{1}{2}\log\left|\frac{z^2-1}{z^2}\right| - \frac{1}{2}\log\frac{3}{4}$$

OR

To evaluate $\int \frac{1}{z(z^2-1)} dx = \int \frac{z}{z^2(z^2-1)} dx = \frac{1}{2}\int \frac{2z}{z^2(z^2-1)} dx$

Put $x^2 = t$.

For each of the differential equations in Exercises 13 to 14, find a particular solution satisfying the given condition:

13. $\cos\left(\frac{dy}{dx}\right) = a \ (a \in R); \ y = 1$ **when** $x = 0$

Sol. The given differential equation is

$$\cos\frac{ay}{az} = a \ (a \in R); \ y = 1 \text{ when } x = 0$$

$$\therefore \frac{ay}{az} = \cos^{-1} a \qquad \Rightarrow dy = (\cos^{-1} a)\, dx$$

Integrating both sides

$$\int 1 \; dy = \int (\cos^{-1} a) \; dx \quad \Rightarrow \quad y = (\cos^{-1} a) \int 1 \; dx$$

$\Rightarrow \quad y = (\cos^{-1} a)\, x + c$ ...(*i*)

To find *c* for particular solution

$y = 1$ when $x = 0$ (given) $\quad \therefore$ From (*i*), $1 = c$.

Putting $c = 1$ in (*i*), $y = x \cos^{-1} a + 1$

$\Rightarrow \quad y - 1 = x \cos^{-1} a \quad \Rightarrow \quad \frac{y - 1}{z} = \cos^{-1} a$

$\Rightarrow \cos\left(\frac{y-1}{z}\right) = a$ which is the required particular solution.

14. $\frac{dy}{dx}$ **= y tan x; y = 1 when x = 0**

Sol. The given differential equation is $\frac{ay}{az} = y \tan x$

$\Rightarrow dy = y \tan x \, dx$

Separating variables, $\frac{ay}{y} = \tan x \, dx$

Integrating both sides $\int \frac{1}{y} dy = \int \tan z \, dx$

$\Rightarrow \log |y| = \log |\sec x| + \log |c|$

$\Rightarrow \log |y| = \log |c \sec x| \quad \Rightarrow \quad |y| = |c \sec x|$

$\therefore \quad y = \pm c \sec x$

or $\quad y = C \sec x$ ...(*i*)

where $C = \pm c$

To find C for particular solution

Putting $y = 1$ and $x = 0$ in (*i*), $1 = C \sec 0 = C$

Putting $C = 1$ in (*i*), the required particular solution is $y = \sec x$.

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

Sol. The given differential equation is $y' = e^x \sin x$

$\Rightarrow \frac{ay}{az} = e^x \sin x \quad \Rightarrow \quad dy = e^x \sin x \, dx$

Integrating both sides, $\int 1 \, dy = \int e^z \sin z \, dx$

or $\quad y = I + C$ ...(*i*)

where $I = \int \underset{I}{e^z} \underset{II}{\sin z} \, dx$ ...(*ii*)

$\left[\text{Applying Product Æule } \int I.II \, az = I \int II \, az - \int \left(\frac{a}{az}(I) \int II \, az\right) az\right]$

$= e^x(-\cos x) - \int e^z(-\cos z) \, dx$

$\Rightarrow \quad I = -e^x \cos x + \int \underset{I}{e^z} \underset{II}{\cos z} \, dx$

Again applying product rule,

$$I = -e^x \cos x + e^x \sin x - \int e^z \sin z \; dx$$

$\Rightarrow$ $I = e^x(-\cos x + \sin x) - I$ [By (*ii*)]

Transposing $2I = e^x(\sin x - \cos x)$

$\therefore$ $I = \dfrac{e^z}{2}(\sin x - \cos x)$

Putting this value of I in (*i*), the required solution is

$$y = \frac{1}{2} e^x (\sin x - \cos x) + c \qquad ...(iii)$$

To find *c*. Given that required curve (*i*) passes through the point (0, 0).

Putting $x = 0$ and $y = 0$ in (*iii*),

$$0 = \frac{1}{2}(-1) + c \quad \text{or} \quad 0 = \frac{-1}{2} + c \quad \therefore \quad c = \frac{1}{2}$$

Putting $c = \frac{1}{2}$ in (*iii*), the required equation of the curve is

$$y = \frac{1}{2} e^x (\sin x - \cos x) + \frac{1}{2}$$

L.C.M. = 2 $\therefore$ $2y = e^x (\sin x - \cos x) + 1$ or $2y - 1 = e^x(\sin x - \cos x)$

which is the required equation of the curve.

16. For the differential equation $xy \frac{dy}{dx} = (x + 2)(y + 2)$, find the solution curve passing through the point (1, – 1).

Sol. The given differential equation is $xy \frac{ay}{az} = (x + 2)(y + 2)$

$$\Rightarrow \quad xy\, dy = (x + 2)(y + 2)\, dx$$

Separating variables $\frac{y}{y+2} dy = \frac{z+2}{z} dx$

Integrating both sides, $\int \frac{y}{y+2} dy = \int \frac{z+2}{z} dx$

$$\Rightarrow \int \frac{y+2-2}{y+2} dy = \int \left(\frac{z}{z} + \frac{2}{z}\right) dx$$

$$\Rightarrow \int \left(\frac{y+2}{y+2} - \frac{2}{y+2}\right) dy = \int \left(1 + \frac{2}{z}\right) dx$$

$$\Rightarrow \int \left(1 - \frac{2}{y+2}\right) dy = \int \left(1 + \frac{2}{z}\right) dx$$

$\Rightarrow$ $y - 2 \log | y + 2 | = x + 2 \log | x | + c$

$\Rightarrow$ $y - x = \log (y + 2)^2 + \log x^2 + c$ $\qquad | \because \; |x|^2 = x^2$

$\Rightarrow$ $y - x = \log ((y + 2)^2 x^2) + c$ $\qquad ...(i)$

To find *c*. Curve (*i*) passes through the point (1, – 1).

Putting $x = 1$ and $y = -1$ in (*i*), $-1 - 1 = \log (1) + c$

or $-2 = c$ $\qquad (\because \log 1 = 0)$

Putting $c = -2$ in (*i*), the particular solution curve is

$$y - x = \log((y+2)^2 x^2) - 2$$

or $y - x + 2 = \log((y+2)^2 x^2)$.

17. Find the equation of the curve passing through the point (0, – 2) given that at any point (*x*, *y*) on the curve the product of the slope of its tangent and *y*-coordinate of the point is equal to the *x*-coordinate of the point.

Sol. Let P(x, y) be any point on the required curve.

According to the question,

(Slope of the tangent to the curve at P(x, y)) × y = x

$$\Rightarrow \frac{dy}{dx} \cdot y = x \quad \Rightarrow \quad y\, dy = x\, dx$$

Now variables are separated.

Integrating both sides $\int y\, dy = \int x\, dx$ $\quad \therefore \frac{y^2}{2} = \frac{x^2}{2} + c$

Multiplying by L.C.M. = 2, $y^2 = x^2 + 2c$

or $\quad y^2 = x^2 + A$ ...(i)

where A = 2c.

Given: Curve (i) passes through the point (0, – 2).

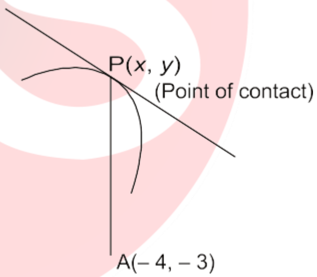
Putting x = 0 and y = – 2 in (i), 4 = A.

Putting A = 4 in (i), equation of required curve is

$$y^2 = x^2 + 4 \quad \text{or} \quad y^2 - x^2 = 4.$$

18. At any point (x, y) of a curve the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, – 3). Find the equation of the curve given that it passes through (– 2, 1).

Sol. According to question, slope of the tangent at any point P(x, y) of the required curve.



= 2 . (Slope of the line joining the point of contact P(x, y) to the given point A(– 4, – 3)).

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{y-(-3)}{x-(-4)}\right) \quad \left|\ \frac{y_2 - y_1}{x_2 - x_1}\right.$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{(x+4)}$$

Cross-multiplying, $(x + 4)\, dy = 2(y + 3)\, dx$

Separating variables, $\frac{dy}{y+3} = \frac{2}{x+4}\, dx$

Integrating both sides, $\int \frac{1}{y+3}\, dy = 2 \int \frac{1}{x+4}\, dx$

$\Rightarrow \log |y + 3| = 2 \log |x + 4| + \log |c|$

(For log | c |, see Foot Note page 612)

$\Rightarrow \log|y+3| = \log|x+4|^2 + \log|c| = \log|c|(x+4)^2$

$\Rightarrow |y+3| = |c|(x+4)^2$

$\Rightarrow y+3 = \pm|c|(x+4)^2$

$\Rightarrow y+3 = C(x+4)^2$ ...(*i*) where $C = \pm|c|$

To find C. Given that curve (i) passes through the point (– 2, 1).
Putting $x = -2$ and $y = 1$ in (i),

$$1 + 3 = C(-2 + 4)^2 \quad \text{or} \quad 4 = 4C \quad \Rightarrow \quad C = \frac{4}{4} = 1.$$

Putting C = 1 in (i), equation of required curve is

$$y + 3 = (x + 4)^2 \quad \text{or} \quad (x + 4)^2 = y + 3.$$

19. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

Sol. Let x be the radius of the spherical balloon at time t.

Given: Rate of change of volume of spherical balloon is constant $= k$ (say)

$$\Rightarrow \frac{a}{at}\left(\frac{4\pi}{3}z^3\right) = k \Rightarrow \frac{4\pi}{3}3x^2\frac{az}{at} = k \Rightarrow 4\pi x^2\frac{az}{at} = k$$

Separating variables, $4\pi x^2\, dx = k\, dt$

Integrating both sides, $4\pi \int z^2\, dx = k \int 1\, dt$

$$\Rightarrow 4\pi\frac{z^3}{3} = kt + c \qquad ...(i)$$

To find c: Given: Initially radius is 3 units.

$\Rightarrow$ When $t = 0$, $x = 3$

Putting $t = 0$ and $x = 3$ in (i), we have

$$\frac{4\pi}{3}(27) = c \quad \text{or} \quad c = 36\pi \qquad ...(ii)$$

To find k: Given: When $t = 3$ sec, $x = 6$ units

Putting $t = 3$ and $x = 6$ in (i), $\frac{4\pi}{3}(6)^3 = 3k + c$.

Putting $c = 36\pi$ from (ii), $\frac{4\pi}{3}(216) = 3k + 36\pi$

or $4\pi(72) - 36\pi = 3k \quad \Rightarrow 288\pi - 36\pi = 3k$

or $3k = 252\pi \quad \Rightarrow k = 84\pi \qquad ...(iii)$

Putting values of c and k from (ii) and (iii) in (i), we have

$$\frac{4\pi}{3}x^3 = 84\pi t + 36\pi$$

Dividing both sides by 4π, $\frac{z^3}{3} = 21t + 9$

$\Rightarrow \quad x^3 = 63t + 27 \qquad \Rightarrow \quad x = (63t + 27)^{1/3}.$

20. In a bank principal increases at the rate of r% per year.Find the value of r if ` 100 double itself in 10 years. ($\log_e 2$ = 0.6931)

Sol. Let P be the principal (amount) at the end of t years.

According to given, rate of increase of principal per year

= r% (of the principal)

$$\Rightarrow \quad \frac{aP}{at} = \frac{r}{100} \times P$$

Separating variables, $\frac{aP}{P} = \frac{r}{100}\, dt$

Integrating both sides, $\log P = \frac{r}{100} t + c$ ...(i)

(Clearly P being principal is > 0, and hence log | P | = log P)

To find *c*. Initial principal = ` 100 (given)

i.e., When $t = 0$, P = 100

Putting $t = 0$ and P = 100 in (i), log 100 = c.

Putting c = log 100 in (i), $\log P = \frac{r}{100} t + \log 100$

$$\Rightarrow \quad \log P - \log 100 = \frac{r}{100} t \quad \Rightarrow \quad \log \frac{P}{100} = \frac{r}{100} t \qquad ...(ii)$$

Putting P = double of itself = 2 × 100 = ` 200

When t = 10 years (given) in (ii),

$$\log \frac{200}{100} = \frac{r}{100} \times 10 \quad \Rightarrow \quad \log 2 = \frac{r}{10}$$

⇒ $r = 10 \log 2 = 10\,(0.6931) = 6.931\%$ (given).

21. **In a bank, principal increases at the rate of 5% per year. An amount of ` 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5}$ = 1.648).**

Sol. Let P be the principal (amount) at the end of t years.

According to given rate of increase of principal per year

= 5% (of the principal)

$$\Rightarrow \quad \frac{aP}{at} = \frac{5}{100} \times P \quad \Rightarrow \quad \frac{aP}{at} = \frac{P}{20}$$

⇒ 20 dP = P dt

Separating variables, $\frac{aP}{P} = \frac{at}{20}$

Integrating both sides, we have

$$\log P = \frac{1}{20} t + c \qquad ...(i)$$

To find *c*. Given: Initial principal deposited with the bank is ` 1000.

⇒ When $t = 0$, P = 1000

Putting $t = 0$ and P = 1000 in (*i*), we have $\log 1000 = c$

Putting $c = \log 1000$ in (*i*), $\log P = \frac{t}{20} + \log 1000$

$\Rightarrow \log P - \log 1000 = \frac{t}{20}$ $\Rightarrow \log \frac{P}{1000} = \frac{t}{20}$

Putting $t = 10$ years (given), we have

$$\log \frac{P}{1000} = \frac{10}{20} = \frac{1}{2} = 0.5$$

$\Rightarrow \quad \frac{P}{1000} = e^{0.5}$ [$\because$ If $\log x = t$, then $x = e^t$]

$\Rightarrow \quad P = 1000\, e^{0.5} = 1000\ (1.648)$ [$\because e^{0.8} = 1.648$ (given)]

$$= 1000 \left(\frac{1648}{1000}\right) = ` \ 1648.$$

22. In a culture the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present.

Sol. Let x be the bacteria present in the culture at time t hours.

According to given,

Rate of **growth** of bacteria is proportional to the number present.

i.e., $\frac{az}{at}$ is proportional to x.

$\therefore \quad \frac{az}{at} = kx$ where k is the constant of proportionality ($k > 0$ because rate of growth (*i.e.*, increase) of bacteria is given.)

$\Rightarrow \quad dx = kx\ dt \qquad \Rightarrow \quad \frac{az}{z} = k\ dt$

Integrating both sides, $\int \frac{1}{z}\, dx = k \int 1\ dt$

$\Rightarrow \quad \log x = kt + c$ ...(*i*)

To find *c*. Given: Initially the bacteria count is x_0 (say) = 1,00,000.

$\Rightarrow$ When $t = 0$, $x = x_0$.

Putting these value in (*i*), $\log x_0 = c$.

Putting $c = \log x_0$ in (*i*), $\log x = kt + \log x_0$

$\Rightarrow \quad \log x - \log x_0 = kt \qquad \Rightarrow \quad \log \frac{z}{z_0} = kt$ ...(*ii*)

To find *k*: According to given, the number of bacteria is increased by 10% in 2 hours.

$\therefore$ Increase in bacteria in 2 hours $= \frac{10}{100} \times 1,00,000 = 10,000$

$\therefore \quad x$, the amount of bacteria at $t = 2$

$= 1,00,000 + 10,000 = 1,10,000 = x_1$ (say)

Putting $x = x_1$ and $t = 2$ in (*ii*)

$$\log \frac{z_1}{z_0} = 2k \quad \Rightarrow k = \frac{1}{2} \log \frac{z_1}{z_0}$$

$$\Rightarrow k = \frac{1}{2} \log \frac{1,10,000}{1,00,000} = \frac{1}{2} \log \frac{11}{10}$$

Putting this value of k in (ii), we have $\log \frac{z}{z_0} = \frac{1}{2}\left(\log \frac{11}{10}\right) t$

When x = 2,00,000 (given);

then $\log \frac{2,00,000}{1,00,000} = \left(\frac{1}{2} \log \frac{11}{10}\right) t \Rightarrow \log 2 = \frac{1}{2} \log \left(\frac{11}{10}\right) t$

Cross-multiplying $2 \log 2 = \left(\log \frac{11}{10}\right) t \Rightarrow t = \frac{2 \log 2}{\left(\log \frac{11}{10}\right)}$ hours.

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = c$ (B) $e^x + e^y = c$ (C) $e^{-x} + e^y = c$ (D) $e^{-x} + e^{-y} = c$

Sol. The given D.E. is $\frac{ay}{az} = e^{x+y}$

$\Rightarrow \frac{ay}{az} = e^x . e^y \qquad \Rightarrow dy = e^x . e^y dx$

Separating variables, $\frac{ay}{(e^y)} = e^x dx$ or $e^{-y} dy = e^x dx$

Integrating both sides $\int e^{-y} dy = \int e^z dx$

$\Rightarrow \frac{e^{-y}}{-1} = e^x + c \Rightarrow -e^{-y} - e^x = c$

Dividing by – 1, $e^{-y} + e^x = -c$

or $e^x + e^{-y}$ = C where C = $-c$ which is the required solution.

$\therefore$ Option (A) is the correct answer.

Exercise 9.5

In each of the Exercises 1 to 5, show that the given differential equation is homogeneous and solve each of them:

1. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$

Sol. The given D.E. is

$$(x^2 + xy)\, dy = (x^2 + y^2)\, dx \qquad \ldots(i)$$

This D.E. looks to be homogeneous as degree of each coefficient of *dx* and *dy* is same throughout (here 2).

From (i), $$\frac{ay}{az} = \frac{z^2 + y^2}{z^2 + zy} = \frac{z^2\left(1 + \frac{y^2}{z^2}\right)}{z^2\left(1 + \frac{y}{z}\right)}$$

or $$\frac{ay}{az} = \frac{1 + \left(\frac{y}{z}\right)^2}{1 + \left(\frac{y}{z}\right)} = F\left(\frac{y}{z}\right) \qquad \ldots(ii)$$

∴ The given D.E. is homogeneous.

Put $\frac{y}{x} = v$**.** Therefore $y = vx$.

∴ $\frac{ay}{az} = v \cdot 1 + x\frac{av}{az} = v + x\frac{av}{az}$

Putting these values of $\frac{y}{z}$ and $\frac{ay}{az}$ in (*ii*), we have

$$v + x\frac{av}{az} = \frac{1+v^2}{1+v}$$

Transposing v to R.H.S., $x\frac{av}{az} = \frac{1+v^2}{1+v} - v$

⇒ $x\frac{av}{az} = \frac{1+v^2-v-v^2}{1+v} = \frac{1-v}{1+v}$

Cross-multiplying $x(1 + v)\, dv = (1 - v)\, dx$

Separating variables $\frac{1+v}{1-v}\, dv = \frac{az}{z}$

Integrating both sides $\int \frac{1+v}{1-v}\, dv = \int \frac{1}{z}\, az$

⇒ $\int \frac{1+1-1+v}{1-v}\, dv = \log x + c$ ⇒ $\int \frac{2-(1-v)}{1-v}\, dv = \log x + c$

⇒ $\int \left(\frac{2}{1-v} - 1\right) dv = \log x + c$ ⇒ $\frac{2\log(1-v)}{-1} - v = \log x + c$

⇒ $-2\log(1-v) - v = \log x + c$

Put $v = \frac{y}{z}$, $-2\log\left(1-\frac{y}{z}\right) - \frac{y}{z} = \log x + c$

Dividing by -1, $2\log\left(\frac{z-y}{z}\right) + \frac{y}{z} = -\log x - c$

⇒ $\log\left(\frac{z-y}{z}\right)^2 + \log x = -\frac{y}{z} - c$ ⇒ $\log\left(\frac{(z-y)^2 z}{z^2}\right) = -\frac{y}{z} - c$

⇒ $\frac{(z-y)^2}{z} = e^{-\frac{y}{z}-c} = e^{-\frac{y}{z}} e^{-c}$ ⇒ $(x-y)^2 = Cx\, e^{-\frac{y}{z}}$ where $C = e^{-c}$

which is the required solution.

2. $y' = \frac{x + y}{x}$

Sol. The given differential equation is $y' = \frac{z + y}{z}$

$$\Rightarrow \quad \frac{ay}{az} = \frac{z}{z} + \frac{y}{z} \qquad \Rightarrow \quad \frac{ay}{az} = 1 + \frac{y}{z} = f\left(\frac{y}{z}\right) \qquad ...(i)$$

$\therefore$ Differential equation (i) is homogeneous.

Put $\frac{y}{z} = v$ $\quad \therefore \quad y = vx$

$\therefore \frac{ay}{az} = v \,.\, 1 + x \frac{av}{az} = v + x \frac{av}{az}$

Putting these values of $\frac{ay}{az}$ and y in (i),

$v + x \frac{av}{az} = 1 + v \quad \Rightarrow \quad x \frac{av}{az} = 1 \quad \Rightarrow \quad x \, dv = dx$

Separating variables, $dv = \frac{az}{z}$

Integrating both sides, $\int 1 \, dv = \int \frac{az}{z}$ $\quad v = \log |x| + c$

Putting $v = \frac{y}{z}$, $\frac{y}{z} = \log |x| + c$ $\quad \therefore \quad y = x \log |x| + cx$

which is the required solution.

3. $(x - y)\, dy - (x + y)\, dx = 0$

Sol. The given differential equation is

$(x - y)\, dy - (x + y)\, dx = 0$ ...(i)

Differential equation (i) looks to be homogeneous because each coefficient of dx and dy is of degree 1.

From (i), $(x - y)\, dy = (x + y)\, dx$

$\therefore \frac{ay}{az} = \frac{z + y}{z - y} = \frac{z\left(1 + \frac{y}{z}\right)}{z\left(1 - \frac{y}{z}\right)}$ or $\frac{ay}{az} = \frac{1 + \frac{y}{z}}{1 - \frac{y}{z}} = f\left(\frac{y}{z}\right)$ ...(ii)

$\therefore$ Differential equation (i) is homogeneous.

Put $\frac{y}{z} = v$ $\quad \therefore \quad y = vx$

$\therefore \frac{ay}{az} = v \,.\, 1 + x \frac{av}{az} = v + x \frac{av}{az}$

Putting these values in (ii), $v + x \frac{av}{az} = \frac{1 + v}{1 - v}$

Shifting v to R.H.S., $x \frac{av}{az} = \frac{1 + v}{1 - v} - v = \frac{1 + v - v + v^2}{1 - v}$

$\frac{av}{} \quad \Rightarrow \quad x \frac{}{az}$

$\frac{}{1+v^2} = 1 - v$

Cross-multiplying, $x(1 - v)\, dv = (1 + v^2)\, dx$

Separating variables, $\frac{(1-v)}{1+v^2}\, dv = \frac{az}{z}$

Integrating both sides, $\int \frac{1-v}{1+v^2}\, dv = \int \frac{1}{z}\, dx + c$

$$\Rightarrow \int \frac{1}{1+v^2}\, dv - \int \frac{v}{1+v^2}\, dv = \int \frac{1}{z}\, dx + c$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2}\int \frac{2v}{1+v^2}\, dv = \log x + c$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + c \qquad \left[\because \int \frac{f'(v)}{f(v)}\, av = \log f(v)\right]$$

Putting $v = \frac{y}{z}$, $\tan^{-1}\frac{y}{z} - \frac{1}{2}\log\left(1+\frac{y^2}{z^2}\right) = \log x + c$

$$\Rightarrow \tan^{-1}\frac{y}{z} - \frac{1}{2}\log\left(\frac{z^2+y^2}{z^2}\right) = \log x + c$$

$$\Rightarrow \tan^{-1}\frac{y}{z} - \frac{1}{2}[\log(x^2+y^2) - \log x^2] = \log x + c$$

$$\Rightarrow \tan^{-1}\frac{y}{z} - \frac{1}{2}\log(x^2+y^2) + \frac{1}{2}\, 2\log x = \log x + c$$

$$\Rightarrow \tan^{-1}\frac{y}{z} - \frac{1}{2}\log(x^2+y^2) = c \Rightarrow \tan^{-1}\frac{y}{z} = \frac{1}{2}\log(x^2+y^2) + c$$

which is the required solution.

4. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

Sol. The given differential equation is

$$(x^2 - y^2)\, dx + 2xy\, dy = 0 \qquad \ldots(i)$$

This differential equation looks to be homogeneous because degree of each coefficient of dx and dy is same (here 2).

From (i), $2xy\, dy = -(x^2 - y^2)\, dx$

$$\Rightarrow \frac{ay}{az} = \frac{-(z^2-y^2)}{2zy} = \frac{y^2-z^2}{2zy}$$

Dividing every term in the numerator and denominator of R.H.S. by x^2,

$$\frac{ay}{az} = \frac{\left(\frac{y}{z}\right)^2 - 1}{2\frac{y}{z}} = f\left(\frac{y}{z}\right) \qquad \ldots(ii)$$

$\therefore$ The given differential equation is homogeneous.

Put $\frac{y}{z} = v$. Therefore $y = vx$ $\therefore$ $\frac{ay}{az} = v \cdot 1 + x\frac{av}{az} = v + x\frac{av}{az}$

Putting these values of z and az in differential equation (*ii*), we have

$$v + x\frac{av}{az} = \frac{v^2 - 1}{2v} \quad \Rightarrow \quad x\frac{av}{az} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow \quad x\frac{av}{az} = \frac{-v^2 - 1}{2v} = -\frac{(v^2 + 1)}{2v} \qquad \therefore\ x\, 2v\, dv = -(v^2 + 1)\, dx$$

$\Rightarrow \quad \dfrac{2v\ av}{v^2+1} = -\dfrac{az}{z}$

Integrating both sides, $\displaystyle\int \frac{2v}{v^2+1}\,dv = -\int \frac{1}{z}\,dx$

$\Rightarrow \quad \log (v^2 + 1) = -\log x + \log c$

$\Rightarrow \quad \log (v^2 + 1) + \log x = \log c$

$\Rightarrow \quad \log (v^2 + 1)\, x = \log c$

$\Rightarrow \quad (v^2 + 1)\, x = c$

Put $v = \dfrac{y}{z}$, $\left(\dfrac{y^2}{z^2}+1\right) x = c$ or $\left(\dfrac{y^2+z^2}{z^2}\right) x = c$

or $\dfrac{y^2+z^2}{z} = c$ or $x^2 + y^2 = cx$

which is the required solution.

5. $x^2 \left(\dfrac{dy}{dx}\right) = x^2 - 2y^2 + xy$

Sol. The given differential equation is $x^2 \dfrac{ay}{az} = x^2 - 2y^2 + xy$

The given differential equation looks to be Homogeneous as all terms in x and y are of same degree (here 2).

Dividing by x^2, $\dfrac{ay}{az} = \dfrac{z^2}{z^2} - \dfrac{2y^2}{z^2} + \dfrac{zy}{z^2}$

or $\dfrac{ay}{az} = 1 - 2\left(\dfrac{y}{z}\right)^2 + \left(\dfrac{y}{z}\right)$ ...(*i*)

$= F\left(\dfrac{y}{z}\right)$

$\therefore$ Differential equation (*i*) is homogeneous.

So put $\dfrac{y}{z} = v$ $\quad\therefore\ y = vx$

$\therefore \quad \dfrac{ay}{az} = v\,.\,1 + x\dfrac{av}{az} = v + x\dfrac{av}{az}$

Putting these values of $\dfrac{y}{z}$ and $\dfrac{ay}{az}$ in (*i*),

$v + x\frac{av}{az} = 1 - 2v^2 + v$ or $x\frac{av}{az} = 1 - 2v^2 \Rightarrow x\,dv = (1 - 2v^2)\,dx$

Separating variables, $\frac{av}{1 - 2v^2} = \frac{az}{z}$

Integrating both sides, $\int \qquad dv = \int \frac{1}{z}\, dx$

$$\log \left| \frac{1+\sqrt{2}v}{} \right| \quad \frac{1}{1^2 - (\sqrt{2}v)^2}$$

$$\Rightarrow \frac{1}{2.1} \quad \frac{1 - \quad 2v}{\sqrt{2} \rightarrow \text{Coefficient of v}} = \log|x| + c$$

$$\left[\because \int \frac{1}{a^2 - z^2} az = \frac{1}{2a} \log \left| \frac{a+z}{a-z} \right| \right]$$

Putting $v = \frac{y}{z}$, $\quad \frac{1}{2\sqrt{2}} \log \left| \frac{1+\sqrt{2}\frac{y}{z}}{1-\sqrt{2}\frac{y}{z}} \right| = \log|x| + c$

Multiplying within logs by x in L.H.S.,

$$\frac{1}{2\sqrt{2}} \log \left| \frac{z+\sqrt{2}y}{z-\sqrt{2}y} \right| = \log|x| + c.$$

In each of the Exercises 6 to 10, show that the given D.E. is homogeneous and solve each of them:

6. $x\,dy - y\,dx = \sqrt{x^2+y^2}\; dx$

Sol. The given differential equation is

$x\,dy - y\,dx = \sqrt{z^2+y^2}\; dx$ or $x\,dy = y\,dx + \sqrt{z^2+y^2}\,.\,dx$

Dividing by dx

$$x\frac{ay}{az} = y + \sqrt{z^2+y^2} \quad \text{or} \quad x\frac{ay}{az} = y + x\sqrt{1+\left(\frac{y}{z}\right)^2}$$

Dividing by x, $\quad \frac{ay}{az} = \frac{y}{z} + \sqrt{1+\left(\frac{y}{z}\right)^2} = F\left(\frac{y}{z}\right) \qquad ...(i)$

$\therefore$ Given differential equation is homogeneous.

Put $\frac{y}{z} = v$ i.e., $y = vx$.

Differentiating w.r.t. x, $\frac{ay}{az} = v + x\frac{av}{az}$

Putting these values of $\frac{y}{z}$ and $\frac{ay}{az}$ in (i), it becomes

$$av \qquad \sqrt{1+v^2} \qquad \frac{av}{} \qquad \sqrt{1+v^2}$$

$$v + x \frac{}{az} = v + \quad \text{or} \quad x \; az =$$

$$\therefore \quad x\, dv = \sqrt{1+v^2}\, dx \quad \text{or} \quad \frac{av}{\sqrt{1+v^2}} = \frac{az}{z}$$

Integrating both sides, $\int \frac{av}{\sqrt{1+v^2}} = \int \frac{az}{z}$

$\therefore \quad \log (v + \sqrt{1+v^2}) = \log x + \log c$

Replacing v by $\frac{y}{z}$, we have

$$\log\left(\frac{y}{z} + \sqrt{1+\frac{y^2}{z^2}}\right) = \log cx \quad \text{or} \quad \frac{y+\sqrt{z^2+y^2}}{z} = cx$$

or $\quad y + \sqrt{z^2+y^2} = cx^2$

which is the required solution.

7. $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y\,dx = \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x\,dy$

Sol. The given D.E. is

$$\left\{z\cos\left(\frac{y}{z}\right) + y\sin\left(\frac{y}{z}\right)\right\} y\,dx = \left\{y\sin\left(\frac{y}{z}\right) - z\cos\left(\frac{y}{z}\right)\right\} x\,dy$$

$$\therefore \quad \frac{ay}{az} = \frac{\left(z\cos\frac{y}{z} + y\sin\frac{y}{z}\right)y}{\left(y\sin\frac{y}{z} - z\cos\frac{y}{z}\right)z} = \frac{zy\cos\frac{y}{z} + y^2\sin\frac{y}{z}}{zy\sin\frac{y}{z} - z^2\cos\frac{y}{z}}$$

Dividing every term in R.H.S. by x^2,

$$\frac{ay}{az} = \frac{\frac{y}{z}\cos\frac{y}{z} + \left(\frac{y}{z}\right)^2\sin\frac{y}{z}}{\frac{y}{z}\sin\frac{y}{z} - \cos\frac{y}{z}} = F\left(\frac{y}{z}\right) \quad \ldots(i)$$

$\therefore$ The given differential equation is homogeneous.

So let us put $\frac{y}{x}$ **=** v**.** Therefore $y = vx$.

$$\therefore \quad \frac{ay}{az} = v \cdot 1 + x\frac{av}{az} = v + x\frac{av}{az}$$

Putting these values in differential equation (i), we have

$$v + x\frac{av}{az} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} \Rightarrow x\frac{av}{az} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$= \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

Cross-multiplying, $x(v \sin v - \cos v)\, dv = 2v \cos v\, dx$

Separating variables, $\dfrac{v\sin v - \cos v}{v\cos v}\, dv = 2\dfrac{dx}{x}$

Integrating both sides, $\displaystyle\int \frac{v\sin v - \cos v}{v\cos v}\, dv = 2\int \frac{1}{x}\, dx$

Using $\dfrac{a-b}{c} = \dfrac{a}{c} - \dfrac{b}{c}$, $\Rightarrow \displaystyle\int \left(\frac{v\sin v}{v\cos v} - \frac{\cos v}{v}\right) dv = 2\int \frac{1}{x}\, dx$

$$\Rightarrow \int \left(\tan v - \frac{1}{v}\right) dv = 2\int \frac{1}{z}\, dx$$

$$\Rightarrow \log|\sec v| - \log|v| = 2\log|x| + \log|c|$$

$$\Rightarrow \log\left|\frac{\sec v}{v}\right| = \log|x|^2 + \log|c| = \log(|c|x^2)$$

$$\Rightarrow \left|\frac{\sec v}{v}\right| = |c|x^2 \qquad \Rightarrow \frac{\sec v}{v} = \pm|c|x^2$$

$$\Rightarrow \sec v = \pm|c|x^2 v$$

Putting $v = \frac{y}{z}$, $\sec\frac{y}{z} = Cx^2\frac{y}{z}$ where $C = \pm|c|$

or $\sec\frac{y}{z} = Cxy \quad \Rightarrow \quad \frac{1}{\cos\frac{y}{z}} = Cxy$

$$\Rightarrow C\,xy\cos\frac{y}{z} = 1 \qquad \Rightarrow xy\cos\frac{y}{z} = \frac{1}{C} = C_1 \text{ (say)}$$

which is the required solution.

8. $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

Sol. The given D.E. is $x\frac{ay}{az} - y + x\sin\left(\frac{y}{z}\right) = 0$

or $x\frac{ay}{az} = y - x\sin\left(\frac{y}{z}\right)$

Dividing every term by x, $\frac{ay}{az} = \frac{y}{z} - \sin\left(\frac{y}{z}\right) = F\left(\frac{y}{z}\right)$ ...(*i*)

Since $\frac{ay}{az} = F\left(\frac{y}{z}\right)$, the given differential equation is homogeneous.

Putting $\frac{y}{z} = v$ *i.e.*, $y = vx$ so that $\frac{ay}{az} = v + x\frac{av}{az}$

Putting these values of $\frac{y}{z}$ and $\frac{ay}{az}$

in (*i*), we have

$$v + x\,\frac{\text{av}}{\text{az}} = v - \sin v$$

or $$x\,\frac{\text{av}}{\text{az}} = -\sin v \qquad \therefore \qquad x\,dv = -\sin v\,dx$$

or $$\frac{\text{av}}{\sin v} = \frac{-\text{az}}{\text{z}} \qquad \text{or} \quad \text{cosec}\,v\,\text{d}v = -\frac{\text{az}}{\text{z}}$$

Integrating, $\log|\text{cosec}\,v - \cot v| = -\log|x| + \log|c|$

or $\log|\text{cosec}\,v - \cot v| = \log\left|\frac{\text{c}}{\text{z}}\right|$

or $\quad \text{cosec } v - \cot v = \pm \dfrac{c}{z}$

Replacing v by $\dfrac{y}{z}$, $\text{cosec } \dfrac{y}{z} - \cot \dfrac{y}{z} = \dfrac{C}{z}$ where $C = \pm c$

$$\Rightarrow \frac{1}{\sin \frac{y}{z}} - \frac{\cos \frac{y}{z}}{\sin \frac{y}{z}} = \frac{C}{z} \Rightarrow \frac{1 - \cos \frac{y}{z}}{\sin \frac{y}{z}} = \frac{C}{z}$$

Cross-multiplying, $x\left(1 - \cos \dfrac{y}{z}\right) = C \sin \dfrac{y}{z}$ which is the required solution.

9. $\boldsymbol{y\,dx + x \log\left(\frac{y}{x}\right) dy - 2x\,dy = 0}$

Sol. The given differential equation is $y\,dx + x\left(\log \dfrac{y}{z}\right) dy - 2x\,dy = 0$

$$\therefore \quad y\,dx = 2x\,dy - x\left(\log \frac{y}{z}\right) dy \quad \text{or} \quad y\,dx = x\left(2 - \log \frac{y}{z}\right) dy$$

$$\therefore \quad \frac{ay}{az} = \frac{\frac{y}{z}}{2 - \log \frac{y}{z}} = F\left(\frac{y}{z}\right) \qquad \ldots(i)$$

Since $\dfrac{ay}{az} = F\left(\dfrac{y}{z}\right)$, the given differential equation is homogeneous.

Putting $\dfrac{y}{z} = v$ *i.e.*, $y = vx$ so that $\dfrac{ay}{az} = v + x\dfrac{av}{az}$

Putting these values of $\dfrac{y}{z}$ and $\dfrac{ay}{az}$ in (*i*), we have

$$v + x\frac{av}{az} = \frac{v}{2 - \log v}$$

$$\text{or} \quad x\frac{av}{az} = \frac{v}{2 - \log v} - v = \frac{v - 2v + v \log v}{2 - \log v} = \frac{-v + v \log v}{2 - \log v}$$

$$\text{or} \quad x\frac{av}{az} = \frac{v(\log v - 1)}{2 - \log v}$$

$$\therefore \qquad x(2 - \log v)\, dv = v\,(\log v - 1)\, dx$$

or $$\frac{2 - \log v}{v(\log v - 1)}\, dv = \frac{az}{z}$$ or $$\frac{1 - (\log v - 1)}{v(\log v - 1)}\, dv = \frac{az}{z}$$

or $$\left[\frac{1}{v(\log v - 1)} - \frac{1}{v}\right] dv = \frac{az}{z}$$

Integrating $\int \left[\frac{1/v}{\log v - 1} - \frac{1}{v}\right] dv = \log|x| + \log|c|$

or $\log|\log v - 1| - \log|v| = \log|x| + \log|c|$ $\left[\because \int \frac{f'(v)}{f(v)} dv = \log|f(v)|\right]$

or $\log\left|\frac{\log v - 1}{v}\right| = \log|cx|$ or $\left|\frac{\log v - 1}{v}\right| = |cx|$

or $\frac{\log v - 1}{v} = \pm cx = Cx$ where $C = \pm c$

or $\log v - 1 = Cx\, v$

Replacing v by $\frac{y}{z}$, we have

$\log\frac{y}{z} - 1 = Cx\left(\frac{y}{z}\right)$ or $\log\frac{y}{z} - 1 = Cy$

which is a primitive (solution) of the given differential equation.

Second solution

The given D.E. is $y\,dx + x\log\left(\frac{y}{x}\right) dy - 2x\,dy = 0$

Dividing every term by dy,

$y\frac{dx}{dy} - x\log\frac{x}{y} - 2x = 0$ $\left[\because \log\frac{y}{x} = \log y - \log x = -(\log x - \log y) = -\log\frac{x}{y}\right]$

Dividing every term by y,

$\frac{dx}{dy} - \frac{x}{y}\log\frac{x}{y} - 2\frac{x}{y} = 0$

$\Rightarrow \frac{dx}{dy} = \frac{x}{y}\log\frac{x}{y} + 2\frac{x}{y}$ ... (i) $\left(= F\left(\frac{x}{y}\right)\right)$

$\therefore$ The given differential is homogeneous.

Put $\frac{x}{y} = v$ i.e. $x = vy$

so that $\frac{dx}{dy} = v + y\frac{dv}{dy}$

Putting these values in D. E. (i), we have

$$v + y\frac{dv}{dy} = v\log v + 2v$$

$$\Rightarrow\quad y\frac{dv}{dy} = v\log v + v = v(\log v + 1)$$

Cross-multiplying $y\, dv = v(\log v + 1)\, dy$

Separating variables $\dfrac{dv}{v(\log v+1)} = \dfrac{dy}{y}$

Integrating both sides $\displaystyle\int \frac{\frac{1}{v}}{\log v+1}\,dv = \int \frac{1}{y}\,dy$

$\therefore \quad \log|\log v+1| = \log|y| + \log|c| = \log|cy| \quad \left[\because \displaystyle\int \frac{f'(v)}{f(v)}\,dv = \log|f(v)|\right]$

$\therefore \quad \log v + 1 = \pm cy = Cy$ where $C = \pm c$

Replacing v by $\dfrac{x}{y}$, we have

$\log \dfrac{x}{y} + 1 = Cy$

or $-\log \dfrac{y}{x} + 1 = Cy \quad \left[\because \log \dfrac{x}{y} = -\log \dfrac{y}{x} \text{ see page 632}\right]$

Dividing by -1, $\log \dfrac{y}{x} - 1 = -Cy$ or $= C_1 y$ which is a primitive (solution) of the given D.E.

10. $\mathbf{(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \dfrac{x}{y}\right) dy = 0}$

Sol. The given differential equation is $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \dfrac{z}{y}\right) dy = 0$

Dividing by dy, $(1 + e^{x/y})\dfrac{az}{ay} + e^{x/y}\left(1 - \dfrac{z}{y}\right) = 0$

or $(1 + e^{x/y})\dfrac{az}{ay} = -e^{x/y}\left(1 - \dfrac{z}{y}\right)$ or $\dfrac{az}{ay} = \dfrac{e^{z/y}\left(\dfrac{z}{y} - 1\right)}{1 + e^{z/y}}$ ...(*i*)

which is a differential equation of the form $\dfrac{az}{ay} = f\left(\dfrac{z}{y}\right)$.

$\therefore$ The given differential equation is homogeneous.

Hence put $\dfrac{x}{y} = v$ *i.e.*, $x = vy$

Differentiating w.r.t. y, $\frac{az}{ay} = v + y \frac{av}{ay}$

Putting these values of $\frac{z}{y}$ and $\frac{az}{ay}$ in (i), we have

$$v + y\frac{av}{ay} = \frac{e^v(v-1)}{1+e^v}.$$

Now transposing v to R.H.S.

$$y\frac{av}{ay} = \frac{ve^v - e^v}{1+e^v} - v = \frac{ve^v - e^v - v - ve^v}{1+e^v} = \frac{-e^v - v}{1+e^v}$$

$\therefore$ $y(1 + e^v)\,dv = -(e^v + v)\,dy$ or $\frac{1+e^v}{v+e^v}\,dv = -\frac{ay}{y}$

Integrating, $\log|(v + e^v)| = -\log|y| + \log|c|$

Replacing v by $\frac{z}{y}$, we have

$$\log\left|\left(\frac{z}{y} + e^{z/y}\right)\right| = \log\left|\frac{c}{y}\right| \quad \text{or} \quad \left|\frac{z}{y} + e^{z/y}\right| = \left|\frac{c}{y}\right|$$

$$\therefore \quad \frac{z}{y} + e^{x/y} = \pm\frac{C}{y}$$

Multiplying every term by y,

$x + y\,e^{x/y} = C$ where $C = \pm c$

which is the required general solution.

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. $(x + y)\,dy + (x - y)\,dx = 0; y = 1$ when $x = 1$ Sol.

The given differential equation is

$(x + y)\,dy + (x - y)\,dx = 0$, $y = 1$ when $x = 1$ ...(*i*)

It looks to be a homogeneous differential equation because each coefficient of dx and dy is of same degree (here 1).

From (*i*), $(x + y)\,dy = -(x - y)\,dx$

$$\therefore \quad \frac{ay}{az} = \frac{-(z-y)}{z+y} = \frac{y-z}{y+z} = \frac{z\left(\frac{y}{z}-1\right)}{z\left(\frac{y}{z}+1\right)}$$

or $$\frac{ay}{az} = \frac{\frac{y}{z}-1}{\frac{y}{z}+1} = f\left(\frac{y}{z}\right) \quad \text{...(ii)}$$

$\therefore$ Given differential equation is homogeneous.

Put $\frac{y}{z} = v$. Therefore $y = vx$.

$\frac{ay}{az} = v\,.\,1 + x$

$$\frac{av}{az} = v + x\frac{av}{az}$$

Putting these values in eqn. (*ii*), $v + x\frac{av}{az} = \frac{v-1}{v+1}$

$$\Rightarrow \quad x\,\frac{av}{az} = \frac{v-1}{v+1} - v = \frac{v-1-v(v+1)}{v+1} = \frac{v-1-v^2-v}{v+1} = \frac{-v^2-1}{v+1}$$

$$\Rightarrow \quad x\,\frac{av}{az} = -\frac{(v^2+1)}{v+1} \qquad \therefore \; x(v+1)\,dv = -(v^2+1)\,dx$$

Separating variables, $\frac{v+1}{v^2+1}\,dv = -\frac{az}{z}$

$$\therefore \quad \int \frac{v}{v^2+1}\,dv + \int \frac{1}{v^2+1}\,dv = -\int \frac{1}{z}\,dx$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2v}{v^2+1}\,dv + \tan^{-1} v = -\log x + c$$

$$\Rightarrow \quad \frac{1}{2}\log(v^2+1) + \tan^{-1} v = -\log x + c \quad \left[\because \int \frac{f'(v)}{f(v)}\,av = \log f(v)\right]$$

Putting $v = \frac{y}{z}$, $\frac{1}{2}\log\left(\frac{y^2}{z^2}+1\right) + \tan^{-1}\frac{y}{z} = -\log x + c$

$$\Rightarrow \quad \frac{1}{2}\log\left(\frac{y^2+z^2}{z^2}\right) + \tan^{-1}\frac{y}{z} = -\log x + c$$

$$\Rightarrow \quad \frac{1}{2}[\log(x^2+y^2) - \log x^2] + \tan^{-1}\frac{y}{z} = -\log x + c$$

$$\Rightarrow \quad \frac{1}{2}\log(x^2+y^2) - \frac{1}{2}\,2\log x + \tan^{-1}\frac{y}{z} = -\log x + c$$

$$\Rightarrow \quad \frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{z} = c \qquad \ldots(iii)$$

To find c: Given: $y = 1$ when $x = 1$.

Putting $x = 1$ and $y = 1$ in (*iii*), $\frac{1}{2}\log 2 + \tan^{-1} 1 = c$

or $c = \frac{1}{2}\log 2 + \frac{\pi}{4}$ $\left(\because \tan\frac{\pi}{4} = 1 \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}\right)$

Putting this value of c in (*iii*),

$$\frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{z} = \frac{1}{2}\log 2 + \frac{\pi}{4}$$

Multiplying by 2,

$$\log (x^2 + y^2) + 2 \tan^{-1} \frac{y}{z} = \log 2 + \frac{\pi}{2}$$

which is the required particular solution.

12. $\boldsymbol{x^2\,dy + (xy + y^2)\,dx = 0;\ y = 1}$ **when** $\boldsymbol{x = 1}$

Sol. The given differential equation is

$$x^2\,dy + (xy + y^2)\,dx = 0 \quad \text{or} \quad x^2\,dy = -y(x + y)\,dx$$

$$\therefore \quad \frac{ay}{az} = -\frac{y(z + y)}{z^2} = -\frac{yz\left(1 + \frac{y}{z}\right)}{z^2}$$

or $\frac{ay}{az} = -\frac{y}{z}\left(1+\frac{y}{z}\right) = F\left(\frac{y}{z}\right)$ ...(*i*)

∴ The given differential equation is homogeneous.

Put $\frac{y}{z} = v$, *i.e.*, $y = vx$

Differentiating w.r.t. *x*, $\frac{ay}{az} = v + x\frac{av}{az}$

Putting these values of $\frac{y}{z}$ and $\frac{ay}{az}$ in differential equation (*i*),

we have $v + x\frac{av}{az} = -v(1+v) = -v - v^2$

Transposing *v* to R.H.S., $x\frac{av}{az} = -v^2 - 2v$

or $x\frac{av}{az} = -v(v+2)$ $x\,dv = -v(v+2)\,dx$

or $\frac{av}{v(v+2)} = -\frac{az}{z}$

Integrating both sides, $\int \frac{1}{v(v+2)}\,dv = -\int \frac{1}{z}\,dx$

or $\frac{1}{2}\int \frac{2}{v(v+2)}\,dv = -\log|x|$ or $\frac{1}{2}\int \frac{(v+2)-v}{v(v+2)}\,dv = -\log|x|$

Separating terms

or $\int\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = -2\log|x|$

or $\log|v| - \log|v+2| = \log x^{-2} + \log|c|$

or $\log\left|\frac{v}{v+2}\right| = \log|cx^{-2}|$

∴ $\left|\frac{v}{v+2}\right| = \left|\frac{c}{z^2}\right|$ ∴ $\frac{v}{v+2} = \pm\frac{c}{z^2}$

Replacing *v* to $\frac{y}{z}$, we have

$\frac{y}{z}$ y ± 2 $= \pm$ z^2 c

or $\dfrac{y}{y+2z} = \pm \dfrac{c}{z^2}$

z

or $x^2y = C(y + 2x)$

where $C = \pm c$ ...(*ii*)

To find C

Put $x = 1$ and $y = 1$ (given) in eqn. (*ii*), $1 = 3C \quad \therefore C = \dfrac{1}{3}$

Putting $C = \dfrac{1}{3}$ in eqn. (*ii*), required particular solution is

$$x^2y = \frac{1}{3}(y + 2x) \quad \text{or} \quad 3x^2y = y + 2x.$$

13. $\left[x \sin^2\left(\frac{y}{x}\right) - y\right]$ ***dx* + *x dy* = 0; *y* =** $\frac{\pi}{4}$ **when *x* = 1**

Sol. The given differential equation is

$$\left(z \sin^2 \frac{y}{z} - y\right) dx + x\,dy = 0;\ y = \frac{\pi}{4},\ x = 1$$

$$\Rightarrow \quad x\,dy = -\left(z \sin^2 \frac{y}{z} - y\right) dx$$

Dividing by dx, $x\frac{ay}{az} = -x \sin^2 \frac{y}{z} + y$

Dividing by x, $\frac{ay}{az} = -\sin^2 \frac{y}{z} + \frac{y}{z}$ ...(*i*)

$$= F\left(\frac{y}{z}\right)$$

$\therefore$ The given differential equation is homogeneous.

Put $\frac{y}{x} = v \quad \therefore \quad y = vx \quad \therefore \quad \frac{ay}{az} = v \cdot 1 + x\frac{av}{az} = v + x\frac{av}{az}$

Putting these values in differential equation (*i*), we have

$$v + x\frac{av}{az} = -\sin^2 v + v \quad \Rightarrow \quad x\frac{av}{az} = -\sin^2 v$$

$$\Rightarrow \quad x\,dv = -\sin^2 v\,dx$$

Separating variables, $\frac{av}{\sin^2 v} = -\frac{az}{z}$

Integrating, $\int \text{cosec}^2 v\,av = -\int \frac{1}{z}\,dx$

$\Rightarrow \quad -\cot v = -\log|x| + c$

Dividing by -1, $\cot v = \log|x| - c$

Putting $v = \frac{y}{z}$, $\cot\frac{y}{z} = \log|x| - c$ ...(*ii*)

To find *c* : $y = \frac{\pi}{4}$ when $x = 1$ (given)

Putting $x = 1$ and $y = \frac{\pi}{4}$

in (*ii*), $\cot \frac{\pi}{4} = \log 1 - c$

or $1 = 0 - c$ or $c = -1$

Putting $c = -1$ in (*ii*), required particular solution is

$$\cot \frac{y}{z} = \log |x| + 1 = \log |x| + \log e = \log |ex|.$$

14. $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$; $y = 0$ **when** $x = 1$

Sol. The given differential equation is

$$\frac{ay}{az} - \frac{y}{z} + \operatorname{cosec} \frac{y}{z} = 0;\ y = 0 \text{ when } x = 1$$

or $\quad \frac{ay}{az} = \frac{y}{z} - \text{cosec}\ \frac{y}{z} = f\left(\frac{y}{z}\right)$ ...(*i*)

$\therefore$ Given differential equation (*i*) is homogeneous.

Put $\frac{y}{z} = v \quad \therefore \quad y = vx \qquad \therefore \quad \frac{ay}{az} = v \,.\, 1 + x\,\frac{av}{az}$

Putting these values in differential equation (*i*),

$$v + x\,\frac{av}{az} = v - \text{cosec}\ v \Rightarrow x\,\frac{av}{az} = \frac{-1}{\sin v}$$

$\therefore \quad x \sin v\ dv = -\ dx$

Separating variables, $\quad \sin v\ dv = -\frac{az}{z}$

Integrating both sides, $\quad \int \sin v\, av = -\int \frac{1}{z}\, dx$

$$-\cos v = -\log|x| + c$$

Dividing by -1, $\quad \cos v = \log|x| - c$

Putting $v = \frac{y}{z}$, $\quad \cos\frac{y}{z} = \log|x| - c$ ...(*ii*)

To find *c*: Given: $y = 0$ when $x = 1$

$\therefore$ From (*ii*), $\cos 0 = \log 1 - c$ or $1 = 0 - c = -c$

$\therefore \quad c = -1$

Putting $c = -1$ in (*ii*), $\cos\frac{y}{z} = \log|x| + 1 = \log|x| + \log e$

$\Rightarrow \cos\frac{y}{z} = \log|ex|$ which is the required particular solution.

15. $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$; $y = 2$ when $x = 1$

Sol. The given differential equation is

$$2xy + y^2 - 2x^2\frac{ay}{az} = 0;\ y = 2 \text{ when } x = 1 \text{ ....(}i\text{)}$$

The given differential equation looks to be homogeneous because each coefficient of *dx* and *dy* is of same degree (2 here).

From (*i*), $-2x^2\frac{ay}{az} = -2xy - y^2 \quad \therefore \quad \frac{ay}{az} = \frac{-2zy}{-2z^2} - \frac{y^2}{-2z^2}$

or $\quad \frac{ay}{az} = \frac{y}{z} + \frac{1}{2}\left(\frac{y}{z}\right)^2 = F\left(y\right)$ ...(*ii*)

$\left(\overline{z} \right)$

$\therefore$ The given differential equation is homogeneous.

Put $\dfrac{y}{z} = v \quad \therefore \quad y = vx \quad \therefore \quad \dfrac{ay}{az} = v \,.\, 1 + x \dfrac{av}{az} = v + x \dfrac{av}{az}$

Putting these values in differential equation (*ii*), we have

$$v + x \frac{av}{az} = v + \frac{1}{2} v^2 \Rightarrow x \frac{av}{az} = \frac{1}{2} v^2 \Rightarrow 2x \; dv = v^2 \; dx$$

Separating variables, $2 \dfrac{av}{v^2} = \dfrac{az}{z}$

Integrating both sides, $2\int v^{-2}\ dv = \int \frac{1}{z}\ dx$

$\Rightarrow\ 2\ \frac{v^{-1}}{-1} = \log|x| + c \quad \Rightarrow\ -\frac{2}{v} = \log|x| + c$

Putting $v = \frac{y}{z}$, $\frac{-2}{\left(\frac{y}{z}\right)} = \log|x| + c$

or $\frac{-2z}{y} = \log|x| + c$ ...(*iii*)

To find *c*: Given: $y = 2$, when $x = 1$.

$\therefore$ From (*iii*), $\frac{-2}{2} = \log 1 + c$ or $-1 = c$

Putting $c = -1$ in (*iii*), the required particular solution is

$-\frac{2z}{y} = \log|x| - 1$

$\Rightarrow\ y(\log|x| - 1) = -2x \quad \Rightarrow\ y = \frac{-2z}{\log IzI - 1}$

$\Rightarrow\ y = \frac{-2z}{-(1 - \log IzI)} \quad \Rightarrow\ y = \frac{2z}{1 - \log IzI}.$

16. Choose the correct answer:
A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution:

(A) $y = vx$ **(B)** $v = yx$ **(C)** $x = vy$ **(D)** $x = v$

Sol. We know that a homogeneous differential equation of the form $\frac{az}{ay} = h\left(\frac{z}{y}\right)$ can be solved by the substitution $\frac{x}{y} = v$ *i.e.*, $x = vy$.

$\therefore$ Option (C) is the correct answer.

17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\ dy - (3y + 2x + 4)\ dx = 0$

(B) $(xy)\ dx - (x^3 + y^3)\ dy = 0$ **(C)** $(x^3 + 2y^2)\ dx + 2xy\ dy = 0$

(D) $y^2\,dx + (x^2 - xy - y^2)\,dy = 0$

Sol. Out of the four given options; option (D) is the only option in which all coefficients of dx and dy are of **same degree** (here 2). It may be noted that xy is a term of second degree.

Hence differential equation in option (D) is **Homogeneous differential equation.**

Exercise 9.6

In each of the following differential equations given in Exercises 1 to 4, find the general solution:

1. $\frac{dy}{dx} + 2y = \sin x$

Sol. The given differential equation is $\frac{ay}{az} + 2y = \sin x$

| Standard form of linear differential equation

Comparing with $\frac{ay}{az} + Py = Q$, we have P = 2 and Q = sin x

$\int P\ dx = \int 2\ dx = 2\int 1\, dx = 2x$ I.F. = $e^{\int P az} = e^{2x}$

Solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\ dx + c$

or $y\, e^{2x} = \int e^{2z} \sin z\ dx + c$

or $y\, e^{2x} = I + c$ ...(*i*)

where $I = \int e^{2z} \sin z\ dx$ ....(*ii*)

I II

Applying Product Rule of Integration

$$\left[\int I.\,II\,az = I\int II\,az - \int\left(\frac{a}{az}(I)\int II\,az\right)az\right],$$

$= e^{2x}(-\cos x) - \int 2e^{2z}(-\cos z)\,dx$

or $I = -e^{2x}\cos x + 2\int e^{2z}\cos x\,dx$

I II

Again applying Product Rule,

$I = -e^{2x}\cos x + 2\left[e^{2z}\sin z - \int 2e^{2z}\sin z\,az\right]$

$\Rightarrow I = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2z}\sin z\,az$

or $I = e^{2x}(-\cos x + 2\sin x) - 4I$

Transposing $5I = e^{2x}(2\sin x - \cos x)$

$\therefore\ I = \frac{e^{2z}}{5}(2\sin x - \cos x)$

Putting this value of I in (*i*), the required solution is

$y\, e^{2x} = \frac{e^{2z}}{5}(2\sin x - \cos x) + c$

Dividing every term by e^{2x}, $y = \frac{1}{5}(2 \sin x - \cos x) + \frac{c}{(e^{2z})}$

or $y = \frac{1}{5}(2 \sin x - \cos x) + c\, e^{-2x}$

which is the required general solution.

2. $\frac{dy}{dx} + 3y = e^{-2x}$

Sol. The given differential equation is $\frac{ay}{az} + 3y = e^{-2x}$

| Standard form of linear differential equation

Comparing with $\frac{ay}{az} + Py = Q$, we have P = 3 and $Q = e^{-2x}$

$\int P\ dx = \int 3\ dx = 3\int 1\ dx = 3x$ I.F. $= e^{\int P\, az} = e^{3x}$

Solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\ dx + c$

or $y\, e^{3x} = \int e^{-2z}\ e^{3x}\ dx + c$ or $= \int e^{-2z+3z}\ dx + c = \int e^{z}\ dx + c$

or $y\, e^{3x} = e^{x} + c$

Dividing every term by e^{3x},

$y = \frac{e^{z}}{e^{3z}} + \frac{c}{e^{3z}}$ or $y = e^{-2x} + ce^{-3x}$

which is the required general solution.

3. $\frac{dy}{dx} + \frac{y}{x} = x^2$

Sol. The given differential equation is $\frac{ay}{az} + \frac{y}{z} = x^2$

It is of the form $\frac{ay}{az} + Py = Q$ Comparing P = $\frac{1}{z}$, $Q = x^2$

$\int P\ dx = \int \frac{1}{z}\ dx = \log x$ $\therefore$ I.F. $= e^{\int P\, az} = e^{\log x} = x$

The general solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\ dx + c$

or $yx = \int z^2 . z\ dx + c = \int z^3\ dx + c$ or $xy = \frac{z^4}{4} + c.$

4. $\frac{dy}{dx}$ + **(sec** ***x*****)** ***y*** **= tan** ***x*** $\left(0 \le x < \frac{\pi}{2}\right)$

Sol. The given differential equation is ay ay

$\frac{dy}{dx} + (\sec x)\, y = \tan x$

It is of the form $\frac{dy}{dx} + Py = Q$.

Comparing $P = \sec x,\ Q = \tan x$

$$\int P\, dx = \int \sec x\, dx = \log(\sec x + \tan x)$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

The general solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + c$

or $y(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)\, dx + c$

$= \int (\sec x \tan x + \tan^2 x)\, dx + c = \int (\sec x \tan x + \sec^2 x - 1)\, dx + c$

$= \sec x + \tan x - x + c$

or $y(\sec x + \tan x) = \sec x + \tan x - x + c$.

For each of the following differential equations given in Exercises 5 to 8, find the general solution:

5. $\cos^2 x \frac{dy}{dx} + y = \tan x \left(0 \le x < \frac{\pi}{2}\right)$

Sol. The given differential equation is $\cos^2 x \frac{ay}{az} + y = \tan x$

Dividing throughout by $\cos^2 x$ to make the coefficient of $\frac{ay}{az}$ unity,

$$\frac{ay}{az} + \frac{y}{\cos^2 z} = \frac{\tan z}{\cos^2 z} \Rightarrow \frac{ay}{az} + (\sec^2 x)\, y = \sec^2 x \tan x$$

It is of the form $\frac{ay}{az} + Py = Q$.

Comparing $P = \sec^2 x$, $Q = \sec^2 x \tan x$

$\int P\, dx = \int \sec^2 z\, dx = \tan x$ I.F. $= e^{\int P az} = e^{\tan x}$

The general solution is $y(\text{I.F.}) = \int Q\, (\text{I.F.})\, dx + c$

or $y e^{\tan x} = \int \sec^2 z \tan x \,.\, e^{\tan x}\, dx + c$ ...(*i*)

Put $\tan x = t$. Differentiating $\sec^2 x\, dx = dt$

$\therefore \int \sec^2 z \tan x\, e^{\tan x}\, dx = \int t\ \ e^t\, dt$

I II

Applying integration by Product Rule,

$= t \,.\, e^t - \int 1 \,.\, e^t\, dt = t \,.\, e^t - e^t = (t - 1)\, e^t = (\tan x - 1)\, e^{\tan x}$

Putting this value in eqn. (*i*), $y e^{\tan x} = (\tan x - 1)\, e^{\tan x} + c$

Dividing every term by $e^{\tan x}$,

$y = (\tan x - 1) + c e^{-\tan x}$ which is the required general solution.

6. $x \frac{dy}{dx} + 2y = x^2 \log x$

Sol. The given differential equation is $x \frac{ay}{az} + 2y = x^2 \log x$

Dividing every term by x $\left(\text{To make coeff. of } \frac{ay}{az} \text{ unity}\right)$

$\frac{ay}{} + \frac{2}{}$ z

| (az

$y = x \log x$ |)

It is of the form $\frac{ay}{az} + Py = Q.$

Comparing $P = \frac{2}{z}$, $Q = x \log x$ $\quad \int P\, dx = 2 \int \frac{1}{z} dx = 2 \log x$

$$\text{I.F.} = e^{\int P\, az} = e^{2 \log x} = e^{\log x^2} = x^2 \quad | \because e^{\log f(x)} = f(x)$$

The general solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + c$

or $\quad yx^2 = \int (z \log z) \,.\, x^2\, dx + c = \int (\log z) \,.\, x^3\, dx + c$

$$= \log x \,.\, \frac{z^4}{4} - \int \frac{1}{z} \,.\, \frac{z^4}{4}\, dx + c \qquad = \frac{z^4}{4} \log x - \frac{1}{4} \int z^3 \, dx + c$$

or $\quad yx^2 = \dfrac{z^4}{4} \log x - \dfrac{z^4}{16} + c.$

Dividing by x^2, $y = \dfrac{z^2}{4} \log x - \dfrac{z^2}{16} + \dfrac{c}{z^2}$

$$y = \frac{z^2}{16}(4 \log x - 1) + \frac{c}{z^2}.$$

7. $\boldsymbol{x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x} \log x}$

Sol. The given differential equation is $x \log x \dfrac{ay}{az} + y = \dfrac{2}{z} \log x$

Dividing every term by $x \log x$ to make the coefficient of $\dfrac{ay}{az}$

unity, $\dfrac{ay}{az} + \dfrac{1}{z \log z} y = \dfrac{2}{z^2}$

Comparing with $\dfrac{ay}{az} + Py = Q$, we have

$$P = \frac{1}{z \log z} \text{ and } Q = \frac{2}{z^2}$$

$$\int P\, dx = \int \frac{1}{z \log z} dx = \int \frac{1/z}{\log z} dx = \log(\log x)$$

$$\left[\because \int \frac{f'(z)}{f(z)} az = \log f(z) \right]$$

$$\text{I.F.} = e^{\int P\, az} = e^{\log(\log x)} = \log x$$

The general solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$

or $\quad y \log x = \displaystyle\int \frac{2}{z^2} \log x\, dx = 2 \int \underset{\text{I}}{(\log z)}\, \underset{\text{II}}{z^{-2}}\, dx + c$

Applying Product Rule of integration,

$$= 2\left[(\log z)\frac{z^{-1}}{-1} - \int \frac{1}{z} \,.\, \frac{z^{-1}}{-1} az\right] + c = 2\left[-\frac{\log z}{z} + \int z^{-2} az\right] + c$$

$$= 2\left[-\frac{\log z}{z} + \frac{z^{-1}}{-1}\right] + c \quad \text{or} \quad y \log x = \frac{-2}{z}(1 + \log x) + c.$$

8. $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx\ (x \neq 0)$

Sol. The given differential equation is $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$

Dividing every term by dx, $(1 + x^2)\dfrac{ay}{az} + 2xy = \cot x$

Dividing every term by $(1 + x^2)$ to make coefficient of $\dfrac{ay}{az}$ unity,

$$\frac{ay}{az} + \frac{2z}{1+z^2}\, y = \frac{\cot z}{1+z^2}$$

Comparing with $\frac{ay}{az} + Py = Q$, we have

$P = \frac{2z}{1+z^2}$ and $Q = \frac{\cot z}{1+z^2}$

$$\int P\, dx = \int \frac{2z}{1+z^2}\, dx = \log |\, 1 + x^2 \,| \qquad \left[\because \int \frac{f'(z)}{f(z)} az = \log|f(z)| \right]$$

$= \log (1 + x^2)$ $\qquad [\because 1 + x^2 > 0 \Rightarrow |\, 1 + x^2 \,| = 1 + x^2]$

$$\text{I.F.} = e^{\int P\, az} = e^{\log (1 + x^2)} = 1 + x^2$$

Solution is $\qquad y(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + c$

$\Rightarrow y(1 + x^2) = \int \frac{\cot z}{1+z^2} (1 + x^2)\, dx + c$

$\Rightarrow y(1 + x^2) = \int \cot z + c \Rightarrow y(1 + x^2) = \log |\sin x| + c$

Dividing by $1 + x^2$, $\qquad y = \frac{\log|\sin z|}{1+z^2} + \frac{c}{1+z^2}$

or $y = (1 + x^2)^{-1} \log |\sin x| + c\,(1 + x^2)^{-1}$

which is the required general solution.

For each of the differential equations in Exercises 9 to 12, find the general solution:

9. $x\frac{dy}{dx} + y - x + xy \cot x = 0, \; (x \neq 0)$

Sol. The given differential equation is

$$x\frac{ay}{az} + y - x + xy \cot x = 0$$

$$\Rightarrow \quad x\frac{ay}{az} + y + xy \cot x = x$$

$$\Rightarrow \quad x\frac{ay}{az} + (1 + x \cot x)\, y = x$$

Dividing every term by x to make coefficient of $\frac{ay}{az}$ unity,

$$\frac{ay}{} + \frac{(1 + z \cot z)}{} \quad az \qquad z$$

$y = 1$

Comparing with $\frac{ay}{az} + Py = Q$, we have

$$P = \frac{1 + z\cot z}{z} \quad \text{and} \quad Q = 1$$

$$\int P\,az = \int \frac{(1 + z\cot z)}{z}\,dx = \int \left(\frac{1}{z} + \frac{z\cot z}{z}\right) dx = \int \left(\frac{1}{z} + \cot z\right) dx$$

$$\Rightarrow \quad \int P\,az = \log x + \log \sin x = \log (x \sin x)$$

$$\text{I.F.} = e^{\int P\,az} = e^{\log(x \sin x)} = x \sin x$$

Solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\,az + c$

or $y(x \sin x) = \int \underset{I}{z} . \underset{II}{\sin z}\; dx + c$

$$\left(\text{Applying Product Æule}, \int \text{I . II}\,az = \text{I} \int \text{II}\,az - \int \left(\frac{a}{az}(\text{I}) \int \text{II}\,az\right) az\right)$$

$\Rightarrow$ $y(x \sin x) = x(-\cos x) - \int 1(-\cos z)\; dx + c$

$$= -x \cos x + \int \cos z\,az + c$$

or $y(x \sin x) = -x \cos x + \sin x + c$

Dividing by $x \sin x$, $y = \frac{-z\cos z}{z\sin z} + \frac{\sin z}{z \sin z} + \frac{c}{z \sin z}$

or $$y = -\cot x + \frac{1}{z} + \frac{c}{z \sin z}$$

which is the required general solution.

10. $(x + y)\frac{dy}{dx} = 1$

Sol. The given differential equation is

$(x + y)\frac{ay}{az} = 1$ $\Rightarrow$ $dx = (x + y)\,dy$

$\Rightarrow$ $\frac{az}{ay} = x + y$ $\Rightarrow$ $\frac{az}{ay} - x = y$

| Standard form of linear differential equation

Comparing with $\frac{az}{ay} + Px = Q$, we have, $P = -1$ and $Q = y$

$\int P\,ay = \int -1\,ay = -\int 1\,ay = -y$ $\qquad$ I.F. $= e^{\int P\,ay} = e^{-y}$

$\therefore$ Solution is $x(\text{I.F.}) = \int Q(\text{I.F.})\,ay + c$

or $xe^{-y} = \int ye^{-y}$ ...

$dy + c$

$\left(\text{Applying Product Æule, } \int \mathrm{I} \,.\, \mathrm{II}\; ay = \mathrm{I} \int \mathrm{II}\; ay - \int \left(\frac{a}{ay} (\mathrm{I}) \int \mathrm{II}\; ay \right) ay \right)$

$$\Rightarrow xe^{-y} = y \frac{e^{-y}}{-1} - \int 1 . \frac{e^{-y}}{-1} dy + c \quad = -ye^{-y} + \int e^{-y} dy + c$$

$$= -ye^{-y} + \frac{e^{-y}}{-1} + c$$

$$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + c$$

Dividing every term by e^{-y}, $x = -y - 1 + \frac{c}{(e^{-y})}$

or $x + y + 1 = ce^{y}$

which is the required general solution.

11. $y\,dx + (x - y^2)\,dy = 0$

Sol. The given differential equation is $y\,dx + (x - y^2)\,dy = 0$

Dividing by dy, $y\frac{az}{ay} + x - y^2 = 0$ or $y\frac{az}{ay} + x = y^2$

Dividing every term by y (to make coefficient of $\frac{az}{ay}$ unity),

$\frac{az}{ay} + \frac{1}{y}x = y$ | Standard form of linear differential equation

Comparing with $\frac{az}{ay} + Px = Q$, we have

$P = \frac{1}{y}$ and $Q = y$

$\int P\,ay = \int \frac{1}{y}\,dy = \log y$

$\text{I.F.} = e^{\int P\,ay} = e^{\log y} = y$

Solution is $x(\text{I.F.}) = \int Q(\text{I.F.})\,ay + c$

$\Rightarrow x\,.\,y = \int yy\,ay + c \Rightarrow xy = \int y^2\,ay + c \Rightarrow xy = \frac{y^3}{3} + c$

Dividing by y, $x = \frac{y^2}{3} + \frac{c}{y}$

which is the required general solution.

12. $(x + 3y^2)\frac{dy}{dx} = y\ (y > 0)$

Sol. The given differential equation is $(x + 3y^2)\frac{ay}{az} = y$

$\Rightarrow y\,dx = (x + 3y^2)\,dy \Rightarrow y\frac{az}{ay} = x + 3y^2 \Rightarrow y\frac{az}{ay} - x = 3y^2$

Dividing every term by y (to make coefficient of $\frac{az}{ay}$ unity),

$\frac{az}{ay} - \frac{1}{y}x = 3y$ | Standard form of linear differential equation

Comparing with $\frac{az}{ay} + Px = Q$, we have $P = \frac{-1}{y}$ and $Q = 3y$

$\int P\,ay = -\int \frac{1}{y}\,dy = -\log y = (-1)\log y = \log y^{-1}$

I.F. = $e^{\int P\,ay} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$

Solution is $x(\text{I.F.}) = \int Q(\text{I.F.})\, ay + c$

$$\Rightarrow x \cdot \frac{1}{y} = \int 3y \cdot \frac{1}{y}\, dy + c \Rightarrow \frac{z}{y} = 3\int 1\, dy + c = 3y + c$$

Cross – Multiplying, $x = 3y^2 + cy$

which is the required general solution.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\frac{dy}{dx}$ **+ 2*y* tan *x* = sin *x*; *y* = 0 when *x* =** $\frac{\pi}{3}$

Sol. The given differential equation is

$$\frac{ay}{az} + 2y \tan x = \sin x;\ y = 0 \text{ when } x = \frac{\pi}{3}.$$

(It is standard form of linear differential equation)

Comparing with $\frac{ay}{az} + Py = Q$, we have

$P = 2 \tan x$ and $Q = \sin x$

$$\int P\, dx = 2\int \tan z\, dx = 2 \log \sec x = \log (\sec x)^2$$

$(\because n \log m = \log m^n)$

$$\text{I.F.} = e^{\int P az} = e^{\log (\sec x)2} = (\sec x)^2 = \sec^2 x$$

$\therefore$ Solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$

$$\Rightarrow \quad y \sec^2 x = \int \sin z \sec^2 z\, dx + c$$

$$= \int \frac{\sin z}{\cos^2 z}\, dx + c = \int \frac{\sin z}{\cos z \,.\, \cos z}\, dx + c$$

or $$y \sec^2 x = \int \tan z \sec z\, dx + c = \sec x + c$$

$$\Rightarrow \quad \frac{y}{\cos^2 z} = \frac{1}{\cos z} + c$$

Multiplying by L.C.M. $= \cos^2 x$,

$$y = \cos x + c \cos^2 x \quad \ldots(i)$$

To find *c*: $y = 0$ when $x = 3$ (given)

$\therefore$ From (i), $0 = \cos\frac{\pi}{3} + c\cos^2\frac{\pi}{3}$

or $0 = \frac{1}{2} + c\left(\frac{1}{2}\right)^2$ or $0 = \frac{1}{2} + \frac{c}{4}$

$$\Rightarrow \quad \frac{c}{4} = \frac{-1}{2} \qquad\qquad \Rightarrow \quad c = -2$$

Putting $c = -2$ in (i), the required particular solution is

$$y = \cos x - 2\cos^2 x.$$

14. $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{l}{l+}$ **;** $y = 0$ **when** $x = 1$

Sol. The given differential equation is

$$(1 + x^2)\frac{ay}{az} + 2xy = \frac{1}{1+z^2}\ ;\ y = 0 \text{ when } x = 1$$

Dividing every term by $(1 + x^2)$ to make coefficient of $\frac{ay}{az}$ unity,

$$\frac{ay}{az} + \frac{2z}{1+z^2}\ y = \frac{1}{(1+z^2)^2}$$

Comparing with $\frac{ay}{az} + Py = Q$, we have

$$P = \frac{2z}{1+z^2} \quad \text{and} \quad Q = \frac{1}{(1+z^2)^2}$$

$$\int P\,dx = \int \frac{2z}{1+z^2}\,dx = \int \frac{f'(z)}{f(z)}\,dx = \log f(x) = \log(1 + x^2)$$

$$\text{I.F.} = e^{\int P\,az} = e^{\log(1+x^2)} = 1 + x^2$$

Solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$

or $\quad y(1 + x^2) = \int \frac{1}{(1+z^2)^2}(1 + x^2)\,dx + c$

or $\quad y(1 + x^2) = \int \frac{1}{z^2+1}\,dx + c = \tan^{-1}x + c$

or $\quad y(1 + x^2) = \tan^{-1}x + c \qquad \ldots(i)$

To find c: $y = 0$ when $x = 1$

Putting $y = 0$ and $x = 1$ in (i), $0 = \tan^{-1} 1 + c$

or $0 = \frac{\pi}{4} + c$ $\left[\because \tan\frac{\pi}{4} = 1\right]$ $\Rightarrow$ $c = -\frac{\pi}{4}$

Putting $c = -\frac{\pi}{4}$ in (i), required particular solution is

$$y(1 + x^2) = \tan^{-1} x - \frac{\pi}{4}.$$

15. $\frac{dy}{dx}$ **– 3*y* cot *x* = sin 2*x* ; *y* = 2 when *x* =** $\frac{\pi}{2}$

Sol. The given differential equation is $\frac{ay}{az} - 3y \cot x = \sin 2x$

Comparing with $\frac{ay}{az} + Py = Q$, we have

$$P = -3 \cot x \text{ and } Q = \sin 2x$$

$$\int P\, dx = -3 \int \cot z \; dx = -3 \log \sin x = \log (\sin x)^{-3}$$

$$\text{I.F.} = e^{\int P\, az} = e^{\log (\sin x)^{-3}} = (\sin x)^{-3} = \frac{1}{\sin^3 z}$$

The general solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\; dx + c$

or $y \frac{1}{\sin^3 z} = \int \sin 2z \,.\, \frac{1}{\sin^3 z}\; dx + c$

or $\frac{y}{\sin^3 z} = \int \frac{2 \sin z \cos z}{\sin^3 z}\; dx + c \quad = 2 \int \frac{\cos z}{\sin^2 z}\; dx + c$

$= 2 \int \frac{\cos z}{\sin z \,.\, \sin z} dx + c \quad = 2 \int \operatorname{cosec} z \cot z \; dx = -2 \operatorname{cosec} x + c$

or $\frac{y}{\sin^3 z} = -\frac{2}{\sin z} + c$

Multiplying every term by L.C.M. $= \sin^3 x$

$$y = -2 \sin^2 x + c \sin^3 x \qquad ...(i)$$

To find *c*: Putting $y = 2$ and $x = \frac{\pi}{2}$ (given) in (*i*),

$2 = -2 \sin^2 \frac{\pi}{2} + c \sin^3 \frac{\pi}{2}$ or $2 = -2 + c$ or $c = 4$

Putting $c = 4$ in (*i*), the required particular solution is

$$y = -2 \sin^2 x + 4 \sin^3 x.$$

16. Find the equation of the curve passing through the origin, given that the slope of the tangent to the curve at any point (*x*, *y*) is equal to the sum of coordinates of that point.

Sol. Given: Slope of the tangent to the curve at any point (x, y) = Sum of coordinates of the point (x, y).

$\Rightarrow \frac{ay}{az} = x + y \qquad \Rightarrow$ ay az

$- y = x$

Comparing with $\frac{ay}{az} + Py = Q$, we have $P = -1$ and $Q = x$

$\int P\ dx = \int -1\ dx = -\int 1\ dx = -x$ I.F. $= e^{\int P\, az} = e^{-x}$

Solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\ dx + c$

i.e., $ye^{-x} = \int \underset{\text{I}}{z}\ \underset{\text{II}}{e^{-z}}\ dx + c$

$\int$

$$\left[\text{Applying Product Æule: } \int \text{I . II az} = \text{I} \int \text{II az} - \int \frac{\text{a}}{\text{az}} (\text{I}) \left(\int \text{II az}\right) \text{az}\right]$$

$$\Rightarrow \quad ye^{-x} = x \frac{e^{-z}}{-1} - \int 1 . \frac{e^{-z}}{-1} \, dx + c$$

or $ye^{-x} = -xe^{-x} + \int e^{-z} \text{ az} + c$ or $ye^{-x} = -xe^{-x} + \frac{e^{-z}}{-1} + c$

or $ye^{-x} = -xe^{-x} - e^{-x} + c$ or $\frac{y}{e^z} = -\frac{z}{e^z} - \frac{1}{e^z} + c$

Multiplying by L.C.M. $= e^x$, $y = -x - 1 + ce^x$ ...(i)

To find c: Given: Curve (i) passes through the origin (0, 0).

Putting $x = 0$ and $y = 0$ in (i), $0 = 0 - 1 + c$

or $-c = -1$ or $c = 1$

Putting $c = 1$ in (i), equation of required curve is

$y = -x - 1 + e^x$ or $x + y + 1 = e^x$.

17. Find the equation of the curve passing through the point (0, 2) given that the sum of the coordinates of any point onthe curve exceeds the magnitude of the slope of the tangentto the curve at that point by 5.

Sol. According to question,

Sum of the coordinates of any point say (x, y) on the curve.

= Magnitude of the slope of the tangent to the curve + 5

↓ (because of exceeds)

i.e., $x + y = \frac{\text{ay}}{\text{az}} + 5$

$$\Rightarrow \quad \frac{\text{ay}}{\text{az}} + 5 = x + y \qquad \Rightarrow \quad \frac{\text{ay}}{\text{az}} - y = x - 5$$

Comparing with $\frac{\text{ay}}{\text{az}} + \text{P}y = \text{Q}$, we have

$\text{P} = -1$ and $\text{Q} = x - 5$

$$\int \text{P} \, dx = \int -1 \, dx = -\int 1 \, dx = -x \qquad \text{I.F.} = e^{\int \text{P az}} = e^{-x}$$

Solution is $y(\text{I.F.}) = \int \text{Q (I.F.)} \, dx + c$

or $$ye^{-x} = \int \underset{\text{I}}{(z - 5)} \underset{\text{II}}{e^{-z}} \, dx + c$$

$$\left[\text{Applying Product Æule: } \int \text{I . II az} = \text{I} \int \text{II az} - \int \frac{\text{a}}{\text{az}} (\text{I}) \left(\int \text{II az}\right) \text{az}\right]$$

r y x e−

az

$= (x - 5)$ $\frac{e^{-z}}{-1}$ $\int \frac{-z}{1.\ -1}$ $\Big|$ $\frac{e}{}$ $dx + c$

or $ye^{-x} = -(x-5)e^{-x} + \int e^{-z}\, dx + c$

or $ye^{-x} = -(x-5)e^{-x} + \frac{e^{-z}}{-1} + c$

or $\frac{y}{(e^x)} = -\frac{(z-5)}{(e^z)} - \frac{1}{(e^z)} + c$

Multiplying both sides by L.C.M. = e^x

$$y = -(x-5) - 1 + ce^x$$

or $y = -x + 5 - 1 + ce^x$ or $x + y = 4 + ce^x$ ...(*i*)

To find *c*: Curve (*i*) passes through the point (0, 2).

Putting $x = 0$ and $y = 2$ in (*i*),

$2 = 4 + ce^0$ or $-2 = c$

Putting $c = -2$ in (*i*), required equation of the curve is

$x + y = 4 - 2e^x$ or $y = 4 - x - 2e^x$.

18. Choose the correct answer:

The integrating factor of the differential equation

$$x\frac{dy}{dx} - y = 2x^2 \text{ is}$$

(A) e^{-x} **(B)** e^{-y} **(C)** $\frac{l}{x}$ **(D)** x

Sol. The given differential equation is $x\frac{ay}{az} - y = 2x^2$

Dividing every term by x to make coefficient of $\frac{ay}{az}$ unity,

$\frac{ay}{az} - \frac{1}{z}y = 2x$ | Standard form of linear differential equation

Comparing with $\frac{ay}{az} + Py = Q$, we have $P = \frac{-1}{z}$ and $Q = 2x$

$\therefore \int P\, dx = \int \frac{-1}{z} dx = -\log x = \log x^{-1}$ [$\because n \log m = \log m^n$]

I.F. = $e^{\int P az} = e^{\log x^{-1}} = x^{-1} = \frac{1}{z}$ [$\because e^{\log f(x)} = f(x)$]

$\therefore$ Option (C) is the correct answer.

19. Choose the correct answer:

The integrating factor of the differential equation

$$(1 - y^2)\frac{dx}{dy} + yx = ay \; (-1 < y < 1)$$

(A) $\frac{l}{y^2 - l}$ (B) $\frac{l}{\sqrt{y^2 - l}}$ (C) $\frac{l}{l - y^2}$ (D) $\frac{l}{\sqrt{l - y^2}}$

Sol. The given differential equation is

$$(1 - y^2)\,\frac{az}{ay} + yx = ay \quad (-1 < y < 1)$$

Dividing every term by $(1 - y^2)$ to make coefficient of $\frac{az}{ay}$ unity,

$$\frac{az}{ay} + \frac{y}{1-y^2}\,x = \frac{ay}{1-y^2}$$

| Standard form of linear differential equation

Comparing with $\frac{az}{ay} + Px = Q$, we have

$$P = \frac{y}{1-y^2} \quad \text{and} \quad Q = \frac{ay}{1-y^2}$$

$$\therefore \quad \int P\, dy = \int \frac{y}{1-y^2}\, dy = \frac{-1}{2}\int \frac{-2y}{1-y^2}\, dy$$

$$= \frac{-1}{2}\log(1-y^2) \qquad \left[\because \int \frac{f'(y)}{f(y)} = \log f(y)\right]$$

$$= \log(1-y^2)^{-1/2}$$

$$\text{I.F.} = e^{\int P\, ay} = e^{\log(1-y^2)^{-1/2}}$$

$$= (1-y^2)^{-1/2} \qquad [\because e^{\log f(x)} = f(x)]$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$\therefore$ Option (D) is the correct answer.

MISCELLANEOUS EXERCISE

1. For each of the differential equations given below, indicate its order and degree (if defined)

(*i*) $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$

(*ii*) $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$

(*iii*) $\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$

Sol. (*i*) The given differential equation is

$$\frac{a^2y}{az^2} + 5x\left(\frac{ay}{az}\right)^2 - 6y = \log x$$

The highest order derivative present in this differential equation is $\frac{a^2y}{az^2}$ and hence order of this differential equation is 2.

The given differential equation is a polynomial equation in derivatives and highest power of the highest order derivative $\frac{a^2y}{az^2}$ is 1.

∴ Order 2, Degree 1.

(ii) The given differential equation is

$$\left(\frac{ay}{az}\right)^3 - 4\left(\frac{ay}{az}\right)^2 + 7y = \sin x.$$

The highest order derivative present in this differential equation is $\frac{ay}{az}$ and hence order of this differential equation is 1.

The given differential equation is a polynomial equation in derivatives and highest power of the highest order derivative $\frac{ay}{az}$ is 3. $\left[\because \text{ of } \left(\frac{ay}{az}\right)^3\right]$

∴ Order 1, Degree 3.

(iii) The given differential equation is $\frac{a^4 y}{az^4} - \sin\left(\frac{a^3 y}{az^3}\right) = 0.$

The highest order derivative present in this differential equation is $\frac{a^4 y}{az^4}$ and hence order of this differential equation is 4.

Degree of this differential equation is not defined because the given differential equation is not a polynomial equation in derivatives

$$\left(\text{because of the presence of term } \sin\left(\frac{a^3 y}{az^3}\right)\right).$$

∴ Order 4 and Degree not defined.

2. For each of the exercises given below verify that the given function (implicit of explicit) is a solution of the corresponding differential equation.

(i) $xy = ae^x + be^{-x} + x^2 : x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

(ii) $y = e^x (a \cos x + b \sin x) : \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

(iii) $y = x \sin 3x : \frac{d^2y}{dx^2} + 9y - 6 \cos 3x = 0$

(iv) $x^2 = 2y^2 \log y : (x^2 + y^2)\frac{dy}{dx} - xy = 0$

Sol. (*i*) The given function is

$$xy = ae^{x} + be^{-x} + x^2 \qquad ...(i)$$

To verify: This given function (*i*) is a solution of differential equation $\frac{a^2y}{az^2} + 2\frac{ay}{az} - xy + x^2 - 2 = 0$ ...(*ii*)

Differentiating both sides of (*i*), w.r.t. *x*,

$$x \frac{ay}{az} + y \cdot 1 = ae^{x} + be^{-x}(-1) + 2x$$

or $$x \frac{ay}{az} + y = ae^{x} - be^{-x} + 2x$$

Again differentiating both sides, w.r.t. x

$$x \frac{a^2y}{az^2} + \frac{ay}{az} \cdot 1 + \frac{ay}{az} = ae^{x} + be^{-x} + 2$$

or $$x \frac{a^2y}{az^2} + 2\frac{ay}{az} = ae^{x} + be^{-x} + 2$$

$\therefore$ Putting $ae^{x} + be^{-x} = xy - x^2$ from (i), in R.H.S., we have

$$x \frac{a^2y}{az^2} + 2\frac{ay}{az} = xy - x^2 + 2$$

or $$x \frac{a^2y}{az^2} + 2\frac{ay}{az} - xy + x^2 - 2 = 0$$

which is same as differential equation (ii).

$\therefore$ Function given by (i) is a solution of D.E. (ii).

(ii) The given function is

$$y = e^{x}(a \cos x + b \sin x) \quad ...(i)$$

To verify: Function given by (i) is a solution of differential equation

$$\frac{a^2}{az^2} y \quad 2 \frac{ay}{az} + 2y = 0 \quad ...(ii)$$

From (i),

$$\frac{ay}{az} = \frac{a}{az} e^{x} \cdot (a \cos x + b \sin x) + e^{x} \frac{a}{az}(a \cos x + b \sin x)$$

or $$\frac{ay}{az} = e^{x}(a \cos x + b \sin x) + e^{x}(-a \sin x + b \cos x)$$

$\Rightarrow$ $$\frac{ay}{az} = y + e^{x}(-a \sin x + b \cos x) \quad ...(iii)$$

(By (i))

$$\therefore \frac{a^2y}{az^2} = \frac{ay}{az} + e^{x}(-a \sin x + b \cos x) + e^{x}(-a \cos x - b \sin x)$$

or $$\frac{a^2y}{az^2} = \frac{ay}{az} + e^{x}(-a \sin x + b \cos x) - e^{x}(a \cos x + b \sin x)$$

or $$\frac{a^2y}{} \quad \frac{ay}{} \quad \Big(\frac{ay}{}$$

$az^2 = az + \left(az \right) - y$ (By (*iii*)) and (By (*i*))

or $\frac{a^2y}{az^2} = 2\frac{ay}{az} - 2y$ or $\frac{a^2y}{az^2} - 2\frac{ay}{az} + 2y = 0$

which is same as given differential equation (*ii*).

∴ Function given by (*i*) is a solution of differential equation (*ii*).

(iii) The given function is

$$y = x \sin 3x \quad ...(i)$$

To verify: Function given by (*i*) is a solution of differential equation

$$\frac{a^2y}{az^2} + 9y - 6 \cos 3x = 0 \quad ...(ii)$$

From (*i*), $\frac{ay}{az} = x \,.\, \cos 3x \,.\, 3 + \sin 3x \,.\, 1$

or $\frac{ay}{az} = 3\,x \cos 3x + \sin 3x$

∴ $\frac{a^2y}{az^2} = 3[x(-\sin 3x)\,3 + \cos 3x \,.\, 1] + (\cos 3x)\,3$

or $\frac{a^2y}{az^2} = -9x \sin 3x + 3 \cos 3x + 3 \cos 3x$

$= -9x \sin 3x + 6 \cos 3x$

$= -9y + 6 \cos 3x$ [By (*i*)]

or $\frac{a^2y}{az^2} + 9y - 6 \cos 3x = 0$

which is same as differential equation (*ii*).

∴ Function given by (*i*) is a solution of differential equation (*ii*).

(iv) The given function is

$$x^2 = 2y^2 \log y \quad ...(i)$$

To verify: Function given by (*i*) is a solution of differential equation

$$(x^2 + y^2)\frac{ay}{az} - xy = 0 \quad ...(ii)$$

Differentiating both sides of (*i*) w.r.t. *x*, we have

$$2x = 2\left[y^2 \,.\, \frac{1}{y}\frac{ay}{az} + (\log y)\, 2y \frac{ay}{az}\right]$$

Dividing by 2, $x = \frac{ay}{az}(y + 2y \log y)$

∴ $\frac{ay}{az} = \frac{z}{y + 2y \log y} = \frac{z}{y(1 + 2 \log y)}$

z^2

Pu tting

$2 \log y = y^2$ from (i),

$$\frac{ay}{az} = \frac{z}{y\left(1 + \frac{z^2}{y^2}\right)} = \frac{z}{y\left(\frac{y^2 + z^2}{y^2}\right)} = \frac{zy^2}{y\,(z^2 + y^2)}$$

$$\Rightarrow \qquad \frac{ay}{az} = \frac{zy}{z^2 + y^2}$$

Cross-multiplying, $(x^2 + y^2)\dfrac{ay}{az} = xy$

or $\qquad (x^2 + y^2)\dfrac{ay}{az} - xy = 0$

which is same as differential equation (*ii*).

∴ Function given by (*i*) is a solution of differential equation (*ii*).

3. Form the differential equation representing the family of curves $(x - a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

Sol. Equation of the given family of curves is

$$(x - a)^2 + 2y^2 = a^2$$

or $\quad x^2 + a^2 - 2ax + 2y^2 = a^2$

or $\quad x^2 - 2ax + 2y^2 = 0$

or $\quad x^2 + 2y^2 = 2ax \qquad ...(i)$

Number of arbitrary constants is one only (a here).

So, we shall differentiate both sides of equation (*i*) only once w.r.t.x.

∴ From (*i*), $\quad 2x + 2 \,.\, 2y\dfrac{ay}{az} = 2a$

or $\quad 2x + 4y\dfrac{ay}{az} = 2a \qquad ...(ii)$

Dividing eqn. (*i*) by eqn. (*ii*) (To eliminate a), we have

$$\frac{z^2 + 2y^2}{2z + 4y\dfrac{ay}{az}} = \frac{2az}{2a} = x$$

Cross-multiplying, $x\left(2z + 4y\dfrac{ay}{az}\right) = x^2 + 2y^2$

or $\quad 2x^2 + 4xy\dfrac{ay}{az} = x^2 + 2y^2 \quad$ or $\quad 4xy\dfrac{ay}{az} = 2y^2 - x^2$

$\Rightarrow \quad \dfrac{ay}{az} = \dfrac{2y^2 - z^2}{4zy}$ which is the required differential equation.

4. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$, where c is a parameter.

Sol. The given differential equation is

$$(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy \qquad ...(i)$$

Here each coefficient of dx and dy is of same degree (Here 3), therefore differential equation (*i*) looks to be homogeneous differential equation.

From (i), $\frac{ay}{az} = \frac{(z^3 - 3zy^2)}{y^3 - 3z^2y}$

Dividing every term in the numerator and denominator of R.H.S. by x^3,

$$\frac{ay}{az} = \frac{1-3\left(\frac{y}{z}\right)^2}{\left(\frac{y}{z}\right)^3 - 3\left(\frac{y}{z}\right)} = f\left(\frac{y}{z}\right) \quad ...(ii)$$

Therefore the given differential equation is homogeneous.

Put $\frac{y}{z} = v$. Therefore $y = vx$. $\therefore \frac{ay}{az} = v \cdot 1 + x\frac{av}{az} = v + x\frac{av}{az}$

Putting these values in eqn. (*ii*),

$$v + x\frac{av}{az} = \frac{1-3v^2}{v^3-3v}$$

$$\therefore \quad x\frac{av}{az} = \frac{1-3v^2}{v^3-3v} - v = \frac{1-3v^2-v^4+3v^2}{v^3-3v} \Rightarrow x\frac{av}{az} = \frac{1-v^4}{v^3-3v}$$

Cross-multiplying, $x(v^3 - 3v)\, dv = (1 - v^4)\, dx$

Separating variables, $\frac{(v^3-3v)}{1-v^4} dv = \frac{az}{z}$

Integrating both sides,

$$\int \frac{v^3-3v}{1-v^4} dv = \int \frac{1}{z} dx = \log x + \log c \quad ...(iii)$$

Let us form partial fractions of

$$\frac{v^3-3v}{1-v^4} = \frac{v^3-3v}{(1-v^2)(1+v^2)} \text{ or } \frac{v^3-3v}{1-v^4} = \frac{v^3-3v}{(1-v)(1+v)(1+v^2)}$$

$$= \frac{A}{1-v} + \frac{B}{1+v} + \frac{Cv+D}{1+v^2} \quad ...(iv)$$

Multiplying both sides of (*iv*) by L.C.M. $= (1-v)(1+v)(1+v^2)$,

$v^3 - 3v = A(1+v)(1+v^2) + B(1-v)(1+v^2) + (Cv + D)(1-v^2)$

$= A(1 + v^2 + v + v^3) + B(1 + v^2 - v - v^3) + Cv - Cv^3 + D - Dv^2$

Comparing coefficients of like powers of v,

v^3 $\quad A - B - C = 1$ ...(*v*)

v^2 $\quad A + B - D = 0$ ...(*vi*)

v $\quad A - B + C = -3$ ...(*vii*)

Constants $\quad A + B + D = 0$ ...(*viii*)

Let us solve eqns. (*v*), (*vi*), (*vii*), (*viii*) for A, B, C, D.

Eqn. (*v*) – eqn. (*vii*) gives, $-2C = 4 \Rightarrow C = \frac{-4}{2} = -2$

Eqn. (*vi*) – eqn. (*viii*) gives, $-2D = 0$ or $D = 0$

Putting C = – 2 in (*v*),

$A - B + 2 = 1 \quad \Rightarrow \quad A - B = -1$ ...(*ix*)]

Putting D = 0 in (*vi*),

$A + B = 0$ ...(*x*)

Adding (*ix*) and (*x*),

$$2A = -1 \quad \Rightarrow \quad A = \frac{-1}{2}$$

From (x), $B = -A = \frac{1}{2}$

Putting values of A, B, C and D in (iv), we have

$$\frac{v^3 - 3v}{1 - v^4} = \frac{\frac{-1}{2}}{1 - v} + \frac{\frac{1}{2}}{1 + v} - \frac{2v}{1 + v^2}$$

$$\therefore \int \frac{v^3 - 3v}{1 - v^4}\,dv = \frac{-1}{2}\,\frac{\log(1 - v)}{-1} + \frac{1}{2}\log(1 + v) - \log(1 + v^2) \quad \left[\because \int \frac{f'(v)}{f(v)}dv = \log f(v)\right]$$

$$= \frac{1}{2}\log(1 - v) + \frac{1}{2}\log(1 + v) - \log(1 + v^2)$$

$$= \frac{1}{2}[\log(1 - v) + \log(1 + v)] - \log(1 + v^2)$$

$$= \frac{1}{2}\log(1 - v)(1 + v) - \log(1 + v^2)$$

$$\Rightarrow \int \frac{v^3 - 3v}{1 - v^4}\,dv = \log(1 - v^2)^{1/2} - \log(1 + v^2) = \log\left(\frac{\sqrt{1 - v^2}}{1 + v^2}\right)$$

Putting this value in eqn. (iii),

$$\log\left(\frac{\sqrt{1 - v^2}}{1 + v^2}\right) = \log xc \qquad \therefore \qquad \frac{\sqrt{1 - v^2}}{1 + v^2} = xc$$

Squaring both sides and cross-multiplying, $1 - v^2 = c^2x^2(1 + v^2)^2$

Putting $v = \frac{y}{z}$, $1 - \frac{y^2}{z^2} = c^2x^2\left(1 + \frac{y^2}{z^2}\right)^2$

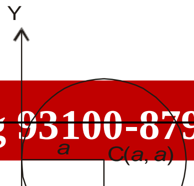
or $\frac{z^2 - y^2}{z^2} = c^2x^2\,\frac{(z^2 + y^2)^2}{z^4}$ or $\frac{z^2 - y^2}{z^2} = \frac{c^2(z^2 + y^2)^2}{z^2}$

or $x^2 - y^2 = C(x^2 + y^2)^2$ where $c^2 = C$

which is the required general solution.

5. Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

Sol. We know that the circle in the first quadrant which touches the co-ordinates axes has centre (a, a) where



a is the radius of the circle. (See adjoining figure)

$\therefore$ Equation of the circle is

$$(x - a)^2 + (y - a)^2 = a^2 \quad ...(i)$$

or $x^2 + y^2 - 2ax - 2ay + a^2 = 0$

Differentiating w.r.t x, we get,

O X

$2x + 2yy' - 2a - 2ay' = 0$

Dividing by 2, $x + yy' = a(1 + y')$

or $$a = \frac{z + yy'}{1 + y'}$$

Substituting the value of a in (i), to eliminate a, we get

$$\left(z - \frac{z + yy'}{1 + y'}\right)^2 + \left(y - \frac{z + yy'}{1 + y'}\right)^2 = \left(\frac{z + yy'}{1 + y'}\right)^2$$

or $$\left(\frac{z + zy' - z - yy'}{1 + y'}\right)^2 + \left(\frac{y + yy' - z - yy'}{1 + y'}\right)^2 = \left(\frac{z + yy'}{1 + y'}\right)^2$$

Multiplying by L.C.M. = $(1 + y')^2$

$$(xy' - yy')^2 + (y - x)^2 = (x + yy')^2 \text{ or}$$
$$y'^2(x - y)^2 + (x - y)^2 = (x + yy')^2 \text{ or}$$
$$(x - y)^2(1 + y'^2) = (x + yy')^2$$

which is the required differential equation.

6. Find the general solution of the differential equation

$$\frac{dy}{dx} + \sqrt{\frac{1 - y^2}{1 - x^2}} = 0.$$

Sol. The given differential equation is

$$\frac{ay}{az} + \sqrt{\frac{1 - y^2}{1 - z^2}} = 0 \Rightarrow \frac{ay}{az} = \frac{-\sqrt{1 - y^2}}{\sqrt{1 - z^2}}$$

$$\Rightarrow \sqrt{1 - z^2}\, dy = -\sqrt{1 - y^2}\, dx$$

Separating Variables, $$\frac{ay}{\sqrt{1 - y^2}} = \frac{-az}{\sqrt{1 - z^2}}$$

Integrating both sides, $$\int \frac{1}{\sqrt{1 - y^2}}\, dy = -\int \frac{1}{\sqrt{1 - z^2}}\, dx$$

$$\Rightarrow \sin^{-1} y = -\sin^{-1} x + c$$
$$\Rightarrow \sin^{-1} x + \sin^{-1} y = c$$

which is the required general solution.

7. Show that the general solution of the differential equation

$$\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0 \text{ is given by}$$

$(x + y + 1) = A(1 - x - y - 2xy)$, where A is parameter.

Sol. The given differential equation is

$$\frac{ay}{az} + \frac{y^2 + y + 1}{z^2 + z + 1} = 0 \Rightarrow \frac{ay}{az} = -\left(\frac{y^2 + y + 1}{z^2 + z + 1}\right)$$

Multiplying by dx and dividing by $y^2 + y + 1$, we have

$$\frac{ay}{y^2 + y + 1} = \frac{-az}{z^2 + z + 1}$$

$$\Rightarrow \quad \frac{ay}{y^2 + y + 1} + \frac{az}{z^2 + z + 1} = 0 \qquad \text{(Variables separated)}$$

Integrating both sides,

$$\int \frac{1}{y^2 + y + 1}\, dy + \int \frac{1}{z^2 + z + 1}\, dx = 0 \qquad \ldots(i)$$

Now, $y^2 + y + 1 = y^2 + y + \frac{1}{4} - \frac{1}{4} + 1$

$$\left[\text{To complete squares, add and subtract } \left(\frac{1}{2} \text{ coeff. of } y\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}\right]$$

$$= \left(y + \frac{1}{2}\right)^2 + \frac{3}{4} = \left(y + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\therefore \int \frac{1}{y^2 + y + 1}\, dy = \int \frac{1}{\left(y + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dy$$

$$\frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1} \frac{y + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2y + 1}{\sqrt{3}}$$

Changing y to x, $\int \frac{1}{z^2 + z + 1}\, dx = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2z + 1}{\sqrt{3}}$

Putting these values in eqn. (i),

$$\frac{2}{\sqrt{3}} \tan^{-1} \frac{2y + 1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2z + 1}{\sqrt{3}} = c$$

Multiplying by $\frac{\sqrt{3}}{2}$, $\quad \tan^{-1} \frac{2y + 1}{\sqrt{3}} + \tan^{-1} \frac{2z + 1}{\sqrt{3}} = \frac{\sqrt{3}}{2} c$

or $\quad \tan^{-1} \dfrac{\frac{2z + 1}{\sqrt{3}} + \frac{2y + 1}{\sqrt{3}}}{1 - \frac{2z + 1}{\sqrt{3}} \cdot \frac{2y + 1}{\sqrt{3}}} = \tan^{-1} c'$

$$\left[\because \tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab} \text{ and replacing } \frac{\sqrt{3}}{2}c \text{ by } \tan^{-1} c'\right]$$

Multiplying every term in the numerator and denominator of L.H.S. by 3, we have

$$\frac{\sqrt{3}\,(2z+2y+2)}{3-(4zy+2z+2y+1)} = c'$$

or $\qquad \sqrt{3}\,(2x + 2y + 2) = c'\,(2 - 2x - 2y - 4xy)$

$\Rightarrow \qquad 2\sqrt{3}\,(x + y + 1) = 2c'(1 - x - y - 2xy)$

Dividing every term by $2\sqrt{3}$, $x + y + 1 = \frac{c'}{\sqrt{3}}(1 - x - y - 2xy)$

or $(x + y + 1) = A(1 - x - y - 2xy)$ where $A = \frac{c'}{\sqrt{3}}$.

8. Find the equation of the curve passing through the point $\left(0, \frac{\pi}{4}\right)$ whose differential equation is

$$\mathbf{\sin x \cos y\, dx + \cos x \sin y\, dy = 0.}$$

Sol. The given differential equation is

$$\sin x \cos y\, dx + \cos x \sin y\, dy = 0$$

$\Rightarrow$ $\sin x \cos y\, dx = -\cos x \sin y\, dy$

Separating variables, $\frac{\sin z}{\cos z} dx = \frac{-\sin y}{\cos y} dy$

$\Rightarrow$ $\tan x\, dx = -\tan y\, dy$

Integrating both sides,

$$\int \tan z\, dx = -\int \tan y\, dy$$

$\Rightarrow$ $\log|\sec x| = -\log|\sec y| + \log|c|$

$\Rightarrow$ $\log|\sec x| + \log|\sec y| = \log|c|$

$\Rightarrow$ $\log|\sec x \sec y| = \log|c|$

$\therefore$ $\sec x \sec y = c$ ...(*i*)

To find *c*: Given: Curve (*i*) passes through $\left(0, \frac{\pi}{4}\right)$.

Putting $x = 0$ and $y = \frac{\pi}{4}$ in (*i*), $\sec 0 \sec \frac{\pi}{4} = c$ or $\sqrt{2} = c$.

Putting $c = \sqrt{2}$ in (*i*), equation of required curve is

$\frac{\sec z}{\cos y} = \sqrt{2}$ $\Rightarrow$ $\cos y = \frac{\sec x}{\sqrt{2}}$ $\Rightarrow$ $\cos y = \frac{\sec z}{\sqrt{2}}$.

9. Find the particular solution of the differential equation $(1 + e^{2x})\, dy + (1 + y^2)\, e^x\, dx = 0$, given that $y = 1$ when $x = 0$.

Sol. The given differential equation is

$$(1 + e^{2x})\, dy + (1 + y^2)\, e^x\, dx = 0$$

Dividing every term by $(1 + y^2)(1 + e^{2x})$, we have

$$\frac{ay}{1 + y^2} + \frac{e^z}{1 + e^{2z}} dx = 0$$

Integrating both sides, we have

$$\int \frac{1}{1+y^2}\,dy + \int \frac{e^z}{1+e^{2z}}\,dx = c$$

or $$\tan^{-1} y + \int \frac{e^z}{1+e^{2z}}\,dx = c \qquad ...(i)$$

To evaluate $\int \frac{e^z}{1+e^{2z}}\,dx$

Put $e^x = t$ $\therefore$ $e^x = \frac{at}{az}$ or $e^x\,dx = dt$

$\therefore \int \frac{e^z\,az}{1+e^{2z}} = \int \frac{at}{1+t^2} = \tan^{-1} t = \tan^{-1} e^x$

Putting this value in (i), $\tan^{-1} y + \tan^{-1} e^x = c$ ...(ii)

To find c: $y = 1$ when $x = 0$ (given)

Putting $x = 0$ and $y = 1$ in (ii), $\tan^{-1} 1 + \tan^{-1} 1 = c$ $\left(\because \overset{0}{e} = 1\right)$

or $\frac{\pi}{4} + \frac{\pi}{4} = c$ $\left[\because \tan\frac{\pi}{4} = 1 \quad \therefore \tan^{-1} 1 = \frac{\pi}{4}\right]$

or $c = \frac{2\pi}{4} = \frac{\pi}{2}$

Putting $c = \frac{\pi}{2}$ in (ii), the particular solution is

$$\tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{2}.$$

10. Solve the differential equation $ye^{x/y}\,dx = (xe^{x/y} + y^2)\,dy$ ($y \neq 0$).

Sol. The given differential equation is

$y \,.\, e^{x/y}\,dx = (x \,.\, e^{x/y} + y^2)\,dy,\ \ y \neq 0$

or $\frac{az}{ay} = \frac{z\,e^{z/y} + y^2}{y\,.\,e^{z/y}} = \frac{z\,e^{z/y}}{y\,e^{z/y}} + \frac{y^2}{y\,e^{z/y}}$

or $\frac{az}{ay} = \frac{z}{y} + y\,e^{-x/y}$ ...(i)

It is not a homogeneous differential equation (because of presence of only y as a factor) yet it can be solved by putting $\frac{z}{y} = v$ *i.e.,* $x = vy$.

so that $\frac{az}{ay} = v + y\frac{av}{ay}$

Putting these values of x and $\frac{az}{ay}$ in (i), we have

$v + y\,\frac{av}{}$ ay

$= v + ye^{-v}$

or $\quad y \frac{av}{ay} = y\, e^{-v} \quad$ or $\quad y \frac{av}{ay} = \frac{y}{e^v}$

Cross-multiplying and dividing both sides by y,

$$e^v\, dv = dy$$

Integrating $\quad e^v = y + c \quad$ or $\quad e^{x/y} = y + c$

which is the required general solution.

11. Find a particular solution of the differential equation $(x - y)(dx + dy) = dx - dy$ given that $y = -1$ when $x = 0$.

Sol. The given differential equation is

$$(x - y)(dx + dy) = dx - dy$$

or $(x - y)\, dx + (x - y)\, dy = dx - dy$

or $(x - y)\, dx - dx + (x - y)\, dy + dy = 0$

or $(x - y - 1)\, dx + (x - y + 1)\, dy = 0$

$\Rightarrow$ $(x - y + 1)\, dy = -(x - y - 1)\, dx$

$\therefore$ $\frac{ay}{az} = -\frac{(z - y - 1)}{z - y + 1}$ ...(*i*)

Put $x - y = t$

Differentiating w.r.t. x, $1 - \frac{ay}{az} = \frac{at}{az}$

$\Rightarrow$ $-\frac{ay}{az} = \frac{at}{az} - 1$ $\Rightarrow$ $\frac{ay}{az} = \frac{-at}{az} + 1$

Putting these values in (*i*), $\frac{-at}{az} + 1 = -\left(\frac{t-1}{t+1}\right)$

$\Rightarrow$ $-\frac{at}{az} = -1 - \left(\frac{t-1}{t+1}\right)$

Multiplying by -1, $\frac{at}{az} = 1 + \frac{t-1}{t+1} = \frac{t+1+t-1}{t+1}$

$\Rightarrow$ $\frac{at}{az} = \frac{2t}{t+1}$

$\Rightarrow$ $(t + 1)\, dt = 2t\, dx$ $\Rightarrow$ $\frac{t+1}{t}\, dt = 2\, dx$

Integrating both sides, $\int \left(\frac{t+1}{t}\right) dt = 2 \int 1\, dx$

or $\int \left(\frac{t}{t} + \frac{1}{t}\right) dt = 2x + c$ or $\int \left(1 + \frac{1}{t}\right) dt = 2x + c$

$\Rightarrow$ $t + \log|t| = 2x + c$

Putting $t = x - y$, $x - y + \log|x - y| = 2x + c$

$\Rightarrow$ $\log|x - y| = x + y + c$ ...(*ii*)

To find *c*: $y = -1$ when $x = 0$

Putting $x = 0$, $y = -1$ in (*ii*),

$\log 1 = 0 - 1 + c$ or $0 = -1 + c$

$\therefore$ $c = 1$

Putting $c = 1$ in (*ii*), required particular solution is

$\log|x - y| = x + y + 1$

$e^{-2\sqrt{x}}$ y dx

12. Solve the differential equation $\left(\frac{}{\sqrt{x}} - \frac{}{\sqrt{x}}\right)\frac{}{dy} = 1 \ (x \neq 0)$

Sol. The given differential equation is

$$\left(\frac{e^{-2\sqrt{z}}}{\sqrt{z}} - \frac{y}{\sqrt{z}}\right)\frac{az}{ay} = 1$$

Multiplying both sides by $\frac{ay}{az}$,

$$\frac{e^{-2\sqrt{z}}}{\sqrt{z}} - \frac{y}{\sqrt{z}} = \frac{ay}{az} \quad \text{or} \quad \frac{ay}{az} + \frac{y}{\sqrt{z}} = \frac{e^{-2\sqrt{z}}}{\sqrt{z}}$$

It is of the form $\frac{ay}{az} + Py = Q.$

Comparing, $P = \frac{1}{\sqrt{z}}$ and $Q = \frac{e^{-2\sqrt{z}}}{\sqrt{z}}$

$$\int P\,dx = \int \frac{1}{\sqrt{z}}\,dx = \int z^{-1/2}\,dx = \frac{z^{1/2}}{1/2} = 2\sqrt{z}$$

$$\text{I.F.} = e^{\int P\,az} = e^{2\sqrt{z}}$$

The general solution is

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$$

or $$ye^{2\sqrt{z}} = \int \frac{e^{-2\sqrt{z}}}{\sqrt{z}} e^{2\sqrt{z}}\,dx + c = \int \frac{1}{\sqrt{z}}\,dx + c$$

or $$y\,.\,e^{2\sqrt{z}} = \int z^{-1/2}\,dx + c = \frac{z^{1/2}}{\frac{1}{2}} + c = 2\sqrt{z} + c$$

Multiplying both sides by $e^{-2\sqrt{z}}$, we have

$y = e^{-2\sqrt{z}}(2\sqrt{z} + c)$ is the required general solution.

13. Find a particular solution of the differential equation $\frac{dx}{dy} + y \cot x = 4x \operatorname{cosec} x \ (x \neq 0)$ given that $y = 0$, when $x = \frac{\pi}{2}$.

Sol. The given differential equation is

$$\frac{ay}{az} + y \cot x = 4x \operatorname{cosec} x$$

(It is standard form of linear differential equation.)

Comparing with $\frac{ay}{az} + Py = Q$, we have

$$P = \cot x \quad \text{and} \quad Q = 4x \operatorname{cosec} x$$

$$\int P\,dx = \text{I.F.} = \int \cot z$$

$e^{\int P\,dz}$ $= \log \sin x$

$= e^{\log \sin x} = \sin x$

Solution is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$

$\Rightarrow \quad y(\sin x) = \int 4z \operatorname{cosec} z \;\sin x\,dx + c$

$\Rightarrow \quad y(\sin x) = 4 \int z \cdot \frac{1}{\sin z} \;\sin x\,dx + c$

or $y \sin x = 4 \int z\, az + c = 4 \cdot \frac{z^2}{2} + c$

or $y \sin x = 2x^2 + c$ ...(*i*)

To find *c*: Given that $y = 0$, when $x = \frac{\pi}{2}$.

Putting $x = \frac{\pi}{2}$ and $y = 0$ in (*i*), $0 = 2 \cdot \frac{\pi^2}{4} + c$

or $0 = \frac{\pi^2}{2} + c \quad \Rightarrow \quad c = \frac{-\pi^2}{2}$

Putting $c = -\frac{\pi^2}{2}$ in (*i*), the required particular solution is

$$y \sin x = 2x^2 - \frac{\pi^2}{2}.$$

14. Find a particular solution of the differential equation $(x + 1)\frac{dy}{dx} = 2e^{-y} - 1$ given that $y = 0$ when $x = 0$.

Sol. The given differential equation is

$$(x + 1)\frac{ay}{az} = 2e^{-y} - 1$$

or $(x + 1)\frac{ay}{az} = \frac{2}{e^y} - 1 = \frac{2 - e^y}{e^y}$

Cross-multiplying, $(x + 1)\, e^y\, dy = (2 - e^y)\, dx$

Separating variables, $\frac{e^y\, ay}{2 - e^y} = \frac{az}{z + 1}$

Integrating both sides, $\int \frac{e^y}{2 - e^y}\, dy = \int \frac{1}{z + 1}\, dx$

Put $e^y = t$. $\therefore$ $e^y = \frac{at}{ay} \Rightarrow e^y\, dy = dt$

$\therefore$ $\int \frac{at}{2 - t} = \log | x + 1 |$

or $\frac{\log | 2 - t |}{-1} = \log | x + 1 | + c$

Putting $t = e^v$, $-\log | 2 - e^y | = \log | x + 1 | + c$

or $\log | x + 1 | + \log | 2 - e^y | = -c$

or $\log | (x + 1)(2 - e^y) | = -c$

or $\quad |(x+1)(2-e^y)| = e^{-c}$

or $\quad (x+1)(2-e^y) = \pm e^{-c}$

or $\quad (x+1)(2-e^y) = C$ where $C = \pm e^{-c}$ ...(*i*)

When $x = 0$, $y = 0$ (given)

$\therefore$ From (*i*), $(1)(2-1) = C$ or $C = 1$

Putting $C = 1$ in (*i*) the required particular solution is

$(x+1)(2-e^y) = 1.$

Note. The particular solution may be written as

$2 - e^y = \frac{1}{z+1}$ or $e^y = 2 - \frac{1}{z+1} = \frac{2z+1}{z+1}$

or $\log e^y = \log\left(\frac{2z+1}{z+1}\right)$ or $y = \log\left(\frac{2z+1}{z+1}\right)$

($\because$ $\log e^y = y \log e = y$ as $\log e = 1$)

which expresses y as an explicit function of x.

15. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what will be the population of the village in 2009?

Sol. Let P be the population of the village at time t.

According to the question, Rate of increase of population of the village is proportional to the number of inhabitants.

$\Rightarrow \frac{dP}{dt} = kP$ (where $k > 0$ because of increase, is the constant of proportionality)

$\Rightarrow dP = kP\, dt \quad \Rightarrow \frac{dP}{P} = k\, dt$

Integrating both sides, $\int \frac{1}{P}\, dP = k \int 1\, dt$

$\Rightarrow \log P = kt + c$ ...(*i*)

To find *c*: Given: Population of the village was P = 20,000 in the year 1999.

Let us take the base year 1999 as $t = 0$.

Putting $t = 0$ and P = 20000 in (*i*), $\log 20000 = c$

Putting $c = \log 20000$ in (*i*), $\log P = kt + \log 20000$

$\therefore \log P - \log 20000 = kt$

$\Rightarrow \log \frac{P}{20000} = kt$ ...(*ii*)

To find *k*: Given: P = 25000 in the year 2004

i.e., when $t = 2004 - 1999 = 5$

Putting P = 25000 and $t = 5$ in (*ii*),

$\log \frac{25000}{20000} = 5k \Rightarrow 5k = \log \frac{5}{4} \Rightarrow k = \frac{1}{5} \log \frac{5}{4}$

Putting $k = \frac{1}{5} \log \frac{5}{4}$ in (*ii*), $\log \frac{P}{20000} = \left(\frac{1}{5} \log \frac{5}{4}\right) t$ ...(*iii*)

To find the population in the year 2009, *i.e.,* when $t = 2009 - 1999 = 10$,

Putting $t = 10$ in (*iii*),

$$\log \frac{P}{20000} = \left(\frac{1}{5}\log\frac{5}{4}\right) \times 10$$

$$= 2\log\frac{5}{4} = \log\left(\frac{5}{4}\right)^2 = \log\frac{25}{16}$$

$$\therefore \frac{P}{20000} = \frac{25}{16}$$

$$\Rightarrow \quad P = \frac{25}{16} \times 20000 = 25 \times 1250 = 31250.$$

16. Choose the correct answer:
The general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$ **is**

(A) $xy = C$ **(B)** $x = Cy^2$ **(C)** $y = Cx$ **(D)** $y = Cx^2$

Sol. The given differential equation is $\frac{y\,az - z\,ay}{y} = 0$

Cross-multiplying, $y\,dx - x\,dy = 0$

$\Rightarrow \quad y\,dx = x\,dy$

Separating variables, $\frac{az}{z} = \frac{ay}{y}$

Integrating both sides, $\log|x| = \log|y| + \log|c|$

$\Rightarrow \log|x| = \log|cy| \quad \Rightarrow |x| = |cy|$

$\Rightarrow \quad x = \pm cy \quad \Rightarrow \quad y = \pm\frac{1}{c}x$

or $y = Cx$ where $C = \pm\frac{1}{c}$

which is the required solution.

$\therefore$ Option (C) is the correct answer.

17. The general solution of a differential equation of the type $\frac{dx}{dy} + P_1x = Q_1$ **is**

(A) $ye^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dy + C$

(B) $y \cdot e^{\int P_1 dx} = \int \left(Q_1 e^{\int P_1 dx}\right) dx + C$

(C) $xe^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dy + C$

(D) $xe^{\int P_1 dx} = \int \left(Q_1 e^{\int P_1 dx}\right) dx + C$

Sol. We know that general solution of differential equation of the type $\frac{az}{ay} + P_1 x = Q_1$ is

$$x \cdot (\text{I.F.}) = \int Q_1 (\text{I.F.})\, dy + c \text{ where I.F.} = e^{\int P_1 ay}$$

$\therefore \quad x\, e^{\int P_1\, ay} = \int (Q_1\, e^{\int P_1\, ay})\, dy + c$

$\therefore$ Option (C) is the correct answer.

18. The general solution of the differential equation

$e^x\, dy + (y\, e^x + 2x)\, dx = 0$ is

(A) $x\, e^y + x^2 = C$ **(B) $x\, e^y + y^2 = C$**

(C) $y\, e^x + x^2 = C$ **(D) $y\, e^y + x^2 = C$**

Sol. The given differential equation is

$$e^x\, dy + (y\, e^x + 2x)\, dx = 0$$

Dividing every term by dx,

$$e^x \frac{ay}{az} + y\, e^x + 2x = 0$$

or $$e^x \frac{ay}{az} + y\, e^x = -2x$$

Dividing every term by e^x to make coefficient of $\frac{ay}{az}$ unity,

$$\frac{ay}{az} + y = \frac{-2z}{e^z}$$ (Standard form of linear differential equation)

Comparing with $\frac{ay}{az} + Py = Q$, we have

$$P = 1 \text{ and } Q = \frac{-2z}{e^z}$$

$$\int P\, dx = \int 1\, dx = x$$

$$\text{I.F.} = e^{\int P\, az} = e^x$$

Solution is y (I.F.) $= \int Q\,(\text{I.F.})\, dx + C$

or $$y\, e^x = \int \frac{-2z}{e^z}\, e^x\, dx + C$$

or $$y\, e^x = -2\int z\, dx + C \quad \text{or} \quad y\, e^x = -2\,\frac{z^2}{2} + C$$

or $$y\, e^x = -x^2 + C \quad \text{or} \quad y\, e^x + x^2 = C$$

$\therefore$ Option (C) is the correct answer.