Exercise 7.1

Find an antiderivative (or integral) of the following functions by the method of inspection in Exercises 1 to 5.

1. **sin $2x$**

Sol. To find an anti derivative of sin $2x$ by Inspection Method.

We know that $\frac{d}{dx}(\cos 2x) = -2 \sin 2x$

Dividing by -2, $\frac{-1}{2}\frac{d}{dx}(\cos 2x) = \sin 2x$

or $\frac{d}{dx}\left(\frac{-1}{2}\cos 2x\right) = \sin 2x$

$\therefore$ By definition; **an** integral or **an** antiderivative of $\sin 2x$ is $\frac{-1}{2}\cos 2x$.

Note. In fact anti derivative or integral of $\sin 2x$ is $\frac{-1}{2}\cos 2x + c$.

For different values of c, we get different antiderivatives. So we omitted c for writing **an** anti derivative.

2. **cos 3*x***

Sol. To find an anti derivative of $\cos 3x$ by Inspection Method.

We know that $\frac{d}{dx}(\sin 3x) = 3 \cos 3x$

Dividing by 3, $\frac{1}{3}\frac{d}{dx}(\sin 3x) = \cos 3x$ or $\frac{d}{dx}\left(\frac{1}{3}\sin 3x\right) = \cos 3x$

$\therefore$ By definition, **an** integral or **an** antiderivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$.

(See note after solution of Q.No.1 for not adding c to the answer.)

3. e^{2x}.

Sol. To find an antiderivative of e^{2x} by Inspection Method.

We know that $\frac{d}{dx}e^{2x} = e^{2x}\frac{d}{dx}(2x) = 2e^{2x}$

Dividing by 2, $\frac{1}{2}\frac{d}{dx}e^{2x} = e^{2x}$ or $\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$

$\therefore$ An antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

4. $(ax + b)^2$.

Sol. To find an anti derivative of $(ax + b)^2$.

We know that $\frac{d}{dx}(ax + b)^3 = 3(ax + b)^2\frac{d}{dx}(ax + b) = 3(ax + b)^2 a$.

Dividing by $3a$, $\frac{1}{3a}\frac{d}{dx}(ax+b)^3 = (ax+b)^2$

or $$\frac{d}{dx}\left[\frac{1}{3a}(ax+b)^3\right] = (ax+b)^2$$

$\therefore$ An anti derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

5. sin $2x - 4e^{3x}$.

Sol. To find an anti derivative of $\sin 2x - 4e^{3x}$ by Inspection Method.

We know that $\frac{d}{dx}(\cos 2x) = -2\sin 2x$

Dividing by -2, $\frac{d}{dx}\left(\frac{-1}{2}\cos 2x\right) = \sin 2x$ ...(*i*)

Again $\frac{d}{dx}e^{3x} = 3e^{3x}$ $\therefore \frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) = e^{3x}$

Multiplying by -4, $\frac{d}{dx}\left(\frac{-4}{3}e^{3x}\right) = -4e^{3x}$ ...(*ii*)

Adding eqns. (*i*) and (*ii*)

$$\frac{d}{dx}\left(\frac{-1}{2}\cos 2x\right) + \frac{d}{dx}\left(\frac{-4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

or $$\frac{d}{dx}\left(\frac{-1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

$\therefore$ An anti derivative of $\sin 2x - 4e^{3x}$ is $\frac{-1}{2}\cos 2x - \frac{4}{3}e^{3x}$.

Evaluate the following integrals in Exercises 6 to 11.

6. $\int (4e^{3x} + 1)\,dx$.

Sol. $\int (4e^{3x}+1)\,dx = \int 4e^{3x}\,dx + \int 1\,dx$

$= 4\int e^{3x}\,dx + x = 4\frac{e^{3x}}{3} + x + c.$ $\left[\because \int e^{ax}\,dx = \frac{e^{ax}}{a} \text{ and } \int 1\,dx = x\right]$

7. $\int x^2\left(1 - \frac{1}{x^2}\right)dx$.

Sol. $\int x^2\left(1 - \frac{1}{x^2}\right)dx = \int\left(x^2 - \frac{x^2}{x^2}\right)dx = \int (x^2 - 1)\,dx$

$= \int x^2\,dx - \int 1\,dx = \frac{x^3}{3} - x + c.$ $\left[\because \int x^n\,dx = \frac{x^{n+1}}{n+1} \text{ if } n \neq -1\right]$

8. $\int (ax^2 + bx + c)\,dx$.

Sol. $\int (ax^2 + bx + c)\,dx = \int ax^2\,dx + \int bx\,dx + \int c\,dx$

$$a\int x^2\ dx + b\int x^1\, dx + c\int 1\ dx \qquad = a\ x^3 + b\ \frac{x^2}{2} + cx + c_1$$

where c_1 is the constant of integration.

9. $\int (2x^2 + e^x)\ dx.$

Sol. $\int (2x^2 + e^x)\ dx = \int 2x^2\ dx + \int e^x\ dx$

$$= 2\int x^2\ dx + \int e^x\ dx \qquad = 2\ \frac{x^{2+1}}{2+1} + e^x + c = 2\ \frac{x^3}{3} + e^x + c.$$

10. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx.$

Sol. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$

Opening the square $= \int \left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x}\frac{1}{\sqrt{x}} \right) dx$

$= \int \left(x + \frac{1}{x} - 2 \right) dx = \int x \, dx + \int \frac{1}{x} dx - \int 2 \, dx$

$= \frac{x^2}{2} + \log|x| - 2x + c.$ $\left[\because \int 2 \, dx = 2 \int 1 \, dx = 2x \right]$

11. $\int \frac{x^3 + 5x^2 - 4}{x^2} dx.$

Sol. $\int \frac{x^3 + 5x^2 - 4}{x^2} dx = \int \left(\frac{x^3}{x^2} + \frac{5x^2}{x^2} - \frac{4}{x^2} \right) dx$

$\left[\text{Using } \frac{a + b - c}{d} = \frac{a}{d} + \frac{b}{d} - \frac{c}{d} \right]$

$= \int (x + 5 - 4x^{-2}) \, dx = \int x^1 \, dx + \int 5 \, dx - \int 4x^{-2} \, dx$

$= \frac{x^2}{2} + 5\int 1 \, dx - 4\int x^{-2} \, dx = \frac{x^2}{2} + 5x - 4\frac{x^{-2+1}}{-2+1} + c$

$= \frac{x^2}{2} + 5x + \frac{4}{x} + c.$

Evaluate the following integrals in Exercises 12 to 16.

12. $\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx.$

Sol. $\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx = \int \left(\frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}} \right) dx$

$= \int (x^{3-1/2} + 3x^{1-1/2} + 4x^{-1/2}) \, dx = \int (x^{5/2} + 3x^{1/2} + 4x^{-1/2}) \, dx$

$= \int x^{5/2} \, dx + 3\int x^{1/2} \, dx + 4\int x^{-1/2} \, dx$

$= \frac{x^{5/2+1}}{\frac{5}{2}+1} + 3\frac{x^{1/2+1}}{\frac{1}{2}+1} + 4\frac{x^{-1/2+1}}{-\frac{1}{2}+1}$

$$+ c = \frac{x^{7/2}}{\frac{7}{2}} + 3\frac{x^{3/2}}{\frac{3}{2}} + 4\frac{x^{1/2}}{\frac{1}{2}} + c$$

$$= \frac{2}{7}x^{7/2} + 2x^{3/2} + 8x^{1/2} + c.$$

13. $\displaystyle\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx.$

Sol. $\displaystyle\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx = \int \frac{x^2(x-1) + (x-1)}{x-1}\, dx$

$$= \int \frac{(x-1)(x^2+1)}{(x-1)}\, dx = \int (x^2+1)\, dx$$

$$= \int x^2 \, dx + \int 1 \, dx = \frac{x^{2+1}}{2+1} + x + c = \frac{x^3}{3} + x + c.$$

14. $\int (1 - x)\sqrt{x} \, dx.$

Sol. $\int (1 - x)\sqrt{x} \, dx = \int (\sqrt{x} - x\sqrt{x}) \, dx$

$$= \int (x^{1/2} - x^1 x^{1/2}) \, dx = \int (x^{1/2} - x^{1+1/2}) \, dx$$

$$= \int (x^{1/2} - x^{3/2}) \, dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{3/2+1}}{\frac{3}{2}+1} + c$$

$$= \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{5/2}}{\frac{5}{2}} + c = \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} + c.$$

15. $\int \sqrt{x} \, (3x^2 + 2x + 3) \, dx.$

Sol. $\int \sqrt{x} \, (3x^2 + 2x + 3) \, dx = \int x^{1/2} (3x^2 + 2x + 3) \, dx$

$$= \int (3x^2 x^{1/2} + 2x \, x^{1/2} + 3x^{1/2}) \, dx = \int (3x^{5/2} + 2x^{3/2} + 3x^{1/2}) \, dx$$

$$\left(\because 2 + \frac{1}{2} = \frac{4+1}{2} = \frac{5}{2}, 1 + \frac{1}{2} = \frac{2+1}{2} = \frac{3}{2}\right)$$

$$= 3\int x^{5/2} \, dx + 2\int x^{3/2} \, dx + 3\int x^{1/2} \, dx$$

$$= 3\frac{x^{5/2+1}}{\frac{5}{2}+1} + 2\frac{x^{3/2+1}}{\frac{3}{2}+1} + 3\frac{x^{1/2+1}}{\frac{1}{2}+1} + c = 3\frac{x^{7/2}}{\frac{7}{2}} + 2\frac{x^{5/2}}{\frac{5}{2}} + 3\frac{x^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{6}{7} x^{7/2} + \frac{4}{5} x^{5/2} + 2x^{3/2} + c.$$

16. $\int (2x - 3 \cos x + e^x) \, dx.$

Sol. $\int (2x - 3 \cos x + e^x) \, dx = \int 2x \, dx - \int 3 \cos x \, dx + \int e^x \, dx$

$= 2\int x^1 \; dx - 3\int \cos x \; dx + \int e^x \, dx = 2\frac{x^2}{2} - 3 \sin x + e^x + c$

$= x^2 - 3 \sin x + e^x + c.$

Evaluate the following integrals in Exercises 17 to 20.

17. $\int (2x^2 - 3 \sin x + 5\sqrt{x}) \; dx.$

Sol. $\int (2x^2 - 3 \sin x + 5\sqrt{x}) \; dx$

$= 2\int x^2 \; dx - 3\int \sin x \; dx + 5\int x^{1/2} \; dx$

$= 2\frac{x^{2+1}}{2+1} - 3(-\cos x) + 5\frac{x^{1/2+1}}{\frac{1}{2}+1} + c = 2\frac{x^3}{3} + 3 \cos x + 5\frac{x^{3/2}}{\frac{3}{2}} + c$

$= 2\frac{x^3}{3} + 3 \cos x + \frac{10}{3} x^{3/2} + c.$

18. $\int \sec x(\sec x + \tan x)\, dx.$

Sol. $\int \sec x(\sec x + \tan x)\, dx = \int (\sec^2 x + \sec x \tan x)\, dx$

$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx = \tan x + \sec x + c.$

19. $\int \frac{\sec^2 x}{\csc^2 x}\, dx.$

Sol. $\int \frac{\sec^2 x}{\csc^2 x}\, dx = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}\, dx = \int \frac{\sin^2 x}{\cos^2 x}\, dx$

$= \int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$(\because \sec^2 x - \tan^2 x = 1 \Rightarrow \sec^2 x - 1 = \tan^2 x)$

$= \int \sec^2 x\, dx - \int 1\, dx = \tan x - x + c.$

Note. Similarly $\int \cot^2 x\, dx = \int (\csc^2 x - 1)\, dx$

$= \int \csc^2 x\, dx - \int 1\, dx = -\cot x - x + c.$

20. $\int \frac{2 - 3\sin x}{\cos^2 x}\, dx.$

Sol. $\int \frac{2 - 3\sin x}{\cos^2 x}\, dx = \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right) dx$

$= \int \left(2\sec^2 x - \frac{3\sin x}{\cos x \cos x}\right) dx = \int (2\sec^2 x - 3\tan x \sec x)\, dx$

$= 2\int \sec^2 x\, dx - 3\int \sec x \tan x\, dx = 2\tan x - 3\sec x + c.$

21. Choose the correct answer:

The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{1/3} + 2x^{1/2} + C$ (B) $\frac{2}{3}x^{2/3} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{3/2} + 2x^{1/2} + C$ (D) $\frac{3}{2}x^{3/2} + \frac{1}{2}x^{1/2} + C.$

Sol. The anti derivative of the $\left(\sqrt{x} + 1\right)$

$$= \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \int \left(x^{1/2} + x^{-1/2}\right) dx$$

$$= \int x^{1/2}\, dx + \int x^{-1/2}\, dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} + \frac{x^{1/2+1}}{\frac{1}{2}+1} + C$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

$\therefore$ Option (C) is the correct answer.

22. Choose the correct answer:

If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ **(B)** $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ **(D)** $x^3 + \frac{1}{x^4} - \frac{129}{8}$.

Sol. Given: $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ and $f(2) = 0$

$\therefore$ By definition of anti derivative (*i.e.*, Integral),

$$f(x) = \int \left(4x^3 - \frac{3}{x^4}\right) dx = 4\int x^3\, dx - 3\int \frac{1}{x^4}\, dx$$

$$= 4 \cdot \frac{x^4}{4} - 3\int x^{-4}\, dx = x^4 - 3\frac{x^{-3}}{-3} + c$$

or $f(x) = x^4 + \frac{1}{(x^3)} + c$ ...(*i*)

To find *c*. Let us make use of $f(2) = 0$ (given)

Putting $x = 2$ on both sides of (*i*),

$f(2) = 16 + \frac{1}{8} + c$ or $0 = \frac{128+1}{8} + c$ ($\because f(2) = 0$ (given))

or $c + \frac{129}{8} = 0$ or $c = \frac{-129}{8}$

Putting $c = \frac{-129}{8}$ in (*i*), $f(x) = x^4 + \frac{1}{(x^3)} - \frac{129}{8}$

$\therefore$ Option (A) is the correct answer

Exercise 7.2

Integrate the functions in Exercises 1 to 8:

1. $\frac{2x}{1+x^2}$

Sol. To evaluate $\int \frac{2x}{1+x^2}\,dx$

Put $1 + x^2 = t$. Therefore $2x = \frac{dt}{dx}$ or $2x\,dx = dt$

$$\therefore \int \frac{2x}{1+x^2}\,dx = \int \frac{dt}{t} = \int \frac{1}{t}\,dt = \log|t| + c$$

Putting $t = 1 + x^2$, $= \log|1 + x^2| + c = \log(1 + x^2) + c$.
($\because$ $1 + x^2 > 0$. Therefore $|1 + x^2| = 1 + x^2$)

2. $\frac{(\log x)^2}{x}$.

Sol. To evaluate $\int \frac{(\log x)^2}{x}\,dx$

Put $\log x = t$. Therefore $\frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{dx}{x} = dt$

$$\therefore \int \frac{(\log x)^2}{x}\, dx = \int t^2\, dt = \frac{t^3}{3} + c$$

Putting $t = \log x$, $= \frac{1}{3}(\log x)^3 + c$.

3. $\frac{1}{x + x\log x}$

Sol. To evaluate $\int \frac{1}{x + x\log x}\, dx = \int \frac{1}{x(1+\log x)}\, dx$

Put $1 + \log x = t$. Therefore $\frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{dx}{x} = dt$

$$\therefore \int \frac{1}{x + x\log x}\, dx = \int \frac{1}{1+\log x}\frac{dx}{x} = \int \frac{1}{t}\, dt = \log|t| + c$$

Putting $t = 1 + \log x$, $\log|1 + \log x| + c$.

4. **$\sin x \sin(\cos x)$**

Sol. To evaluate $\int \sin x \sin(\cos x)\, dx = -\int \sin(\cos x)(-\sin x)\, dx$

Put $\cos x = t$. Therefore $-\sin x = \frac{dt}{dx}$

$\therefore \quad -\sin x\, dx = dt$

$$\therefore \int \sin x \sin(\cos x)\, dx = -\int \sin(\cos x)(-\sin x\, dx)$$

$$= -\int \sin t\, dt = -(-\cos t) + c$$

$$= \cos t + c$$

Putting $t = \cos x$, $= \cos(\cos x) + c$.

5. **$\sin(ax + b)\cos(ax + b)$**

Sol. To evaluate $\int \sin(ax+b)\cos(ax+b)\, dx$

$$= \frac{1}{2}\int 2\sin(ax+b)\cos(ax+b)\, dx = \frac{1}{2}\int \sin 2(ax+b)\, dx$$

$(\because 2\sin\theta\cos\theta = \sin 2\theta)$

$$= \frac{1}{2}\int \sin(2ax+2b)\, dx = \frac{1}{2}\frac{[-\cos(2ax+2b)]}{2a \to \text{Coeff. of } x} + c$$

$= \underline{-1}$ $=$ $4a$

6. $\sqrt{ax+b}$ cos $2(ax + b) + c$.

Sol. To evaluate $\int \sqrt{ax+b}\ dx = \int (ax+b)^{1/2}\ dx$

$$= \frac{(ax+b)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right) a} + c = \frac{(ax+b)^{\frac{3}{2}}}{\frac{3}{2} a} + c \qquad a \to \text{Coeff. of } x$$

$$\left[\because \int (ax+b)^n\ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \text{ if } n \neq -1\right]$$

$= \frac{2}{3a}(ax + b)^{3/2} + c.$

7. $\boldsymbol{x\sqrt{x+2}}$

Sol. To evaluate $\int x\sqrt{x+2}\,dx$

$= \int x\sqrt{x+2}\,dx = \int ((x+2) - 2)\sqrt{x+2}\,dx$

$= \int \left((x+2)(x+2)^{\frac{1}{2}} - 2(x+2)^{\frac{1}{2}}\right) dx = \int \left((x+2)^{\frac{3}{2}} - 2(x+2)^{\frac{1}{2}}\right) dx$

$= \int (x+2)^{\frac{3}{2}}\,dx - 2\int (x+2)^{\frac{1}{2}}\,dx$

$= \frac{(x+2)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right) 1} \to \text{Coeff. of } x - 2\frac{(x+2)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right).1} + c = \frac{(x+2)^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c.$

OR

To evaluate $\int x\sqrt{x+2}\,dx$

Put $\sqrt{\textbf{Linear}}$ **= *t*,** *i.e.,* $\sqrt{x+2} = t.$

Squaring $x + 2 = t^2$ $(\Rightarrow x = t^2 - 2)$

$\therefore \frac{dx}{dt} = 2t,$ *i.e.,* $\frac{dx}{dt} = 2t$ or $dx = 2t\,dt$

$\therefore \int x\sqrt{x+2}\,dx = \int (t^2 - 2)\,t\,.\,2t\,dt = \int 2t^2(t^2 - 2)\,dt$

$= \int 2t^2(t^2 - 2)\,dt = 2\int t^4\,dt - 4\int t^2\,dt = 2\frac{t^5}{5} - 4\frac{t^3}{3} + c$

Putting $t = \sqrt{x+2}$, $= \frac{2}{5}(\sqrt{x+2})^5 - \frac{4}{3}(\sqrt{x+2})^3 + c$

$= \frac{2}{5}((x+2)^{1/2})^5 - \frac{4}{3}((x+2)^{1/2})^3 + c = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c.$

8. $\boldsymbol{x\sqrt{1+2x^2}}$

Sol. To evaluate $\int x\sqrt{1+2x^2}\,dx$

$\sqrt{1+2x^2}$

Let $I = \int x\sqrt{1+2x^2}\ dx = \frac{1}{4}\int \qquad (4x\ dx)$ ...(i)

$\left[\because \frac{d}{dx}(1+2x^2) = 0 + 2 \,.\, 2x = 4x\right]$

Put $1 + 2x^2 = t$. Therefore $4x = \frac{dt}{dx}$ or $4x\ dx = dt$

$\therefore$ From (i), $I = \frac{1}{4}\int \sqrt{t}\ dt = \frac{1}{4}\int t^{1/2}\ dt$

$$= \frac{1}{4}\frac{t^{3/2}}{\frac{3}{2}} + c = \frac{1}{4}\cdot\frac{2}{3}t^{3/2} + c$$

Putting $t = 1 + 2x^2$, $= \frac{1}{6}(1 + 2x^2)^{3/2} + c$.

Integrate the functions in Exercises 9 to 17:

9. $(4x + 2)\sqrt{x^2 + x + 1}$.

Sol. Let $I = \int (4x+2)\sqrt{x^2+x+1}\,dx = \int 2(2x+1)\sqrt{x^2+x+1}\,dx$

$$= \int 2\sqrt{x^2+x+1}\,(2x+1)\,dx \quad \text{...}(i)$$

Put $x^2 + x + 1 = t$. Therefore $(2x + 1) = \frac{dt}{dx}$

$\therefore$ $(2x + 1)\,dx = dt$

$\therefore$ From (i), $I = \int 2\sqrt{t}\,dt = 2\int t^{1/2}\,dt$

$$= 2\frac{t^{3/2}}{\frac{3}{2}} + c = \frac{4}{3}t^{3/2} + c$$

Putting $t = x^2 + x + 1$, $I = \frac{4}{3}(x^2 + x + 1)^{3/2} + c$.

10. $\dfrac{1}{x - \sqrt{x}}$

Sol. Let $I = \int \frac{1}{x - \sqrt{x}}\,dx \quad \text{...}(i)$

Put $\sqrt{\text{Linear}} = t$, *i.e.,* $\sqrt{x} = t$

Squaring $x = t^2$. Therefore $\frac{dx}{dt} = 2t$ or $dx = 2t\,dt$

$\therefore$ From (i), $I = \int \frac{1}{t^2 - t}\,2t\,dt = 2\int \frac{t}{t(t-1)}\,dt$

$$= 2\int \frac{1}{t-1}\,dt = 2\log|t-1| + c \quad \left(\because \int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b|\right)$$

Putting $t = \sqrt{x}$, $I = 2\log|\sqrt{x} - 1| + c$.

11. $\dfrac{x}{\sqrt{x+4}}$, $x > 0$

Sol. Let $I = \int \frac{x}{\sqrt{x+4}}\, dx$ ...(i)

$$= \int \frac{x+4-4}{\sqrt{x+4}}\, dx = \int \left(\frac{x+4}{\sqrt{x+4}} - \frac{4}{\sqrt{x+4}} \right) dx$$

$$= \int \sqrt{x+4}\, dx - 4 \int \frac{1}{\sqrt{x+4}}\, dx \quad \left[\because \frac{t}{\sqrt{t}} = \frac{\sqrt{t}\sqrt{t}}{\sqrt{t}} = \sqrt{t} \right]$$

$$= \int (z+4)^{1/2}\, dx - 4\int (z+4)^{-1/2}\, dx$$

$$= \frac{(z+4)^{3/2}}{\frac{3}{2}(1)} - \frac{4(z+4)^{1/2}}{\frac{1}{2}(1)} + c = \frac{2}{3}(x+4)^{3/2} - 8(x+4)^{1/2} + c$$

$$= \frac{2}{3}(x+4)\sqrt{z+4} - 8\sqrt{z+4} + c$$

$$\left[\because\ t^{3/2} = t^{\frac{2}{2}+\frac{1}{2}} = t^{1+\frac{1}{2}} = t^1 . t^{1/2} = t\sqrt{t}\right]$$

$$= 2\sqrt{z+4}\left(\frac{z+4}{3} - 4\right) + c = 2\sqrt{z+4}\left(\frac{z+4-12}{3}\right) + c$$

$$= \frac{2}{3}\sqrt{z+4}\,(x-8) + c.$$

OR

Put $\sqrt{\text{Linear}} = t$, *i.e.,* $\sqrt{z+4} = t.$

Squaring $x + 4 = t^2 \Rightarrow x = t^2 - 4.$

Therefore $\frac{az}{at} = 2t$ or $dx = 2t\, dt$

$$\therefore\ I = \int \frac{z}{\sqrt{z+4}}\, dx = \int \frac{t^2-4}{t} . 2t\, dt$$

$$= 2\int (t^2-4)\, dt = 2\left[\int t^2\, at - 4\int 1\, at\right]$$

$$= 2\left[\frac{t^3}{3} - 4t\right] + c = \frac{2t}{3}(t^2 - 12) + c.$$

Putting $t = \sqrt{z+4}$, $= \frac{2}{3}\sqrt{z+4}\,(x+4-12) + c$

$$= \frac{2}{3}\sqrt{z+4}\,(x-8) + c.$$

12. $(x^3 - 1)^{1/3} x^5$

Sol. Let $I = \int (z^3-1)^{1/3} x^5\, dx = \int (z^3-1)^{1/3} x^3 x^2\, dx$

$$= \frac{1}{3}\int (z_3 - 1)_{1/3}\, z_3\, (3x^2\, dx) \quad \ldots(i) \qquad \left[\because \frac{a}{az}(z^3 - 1) = 3z^2\right]$$

Put $x^3 - 1 = t$ $\Rightarrow$ $x^3 = t + 1$

$\therefore \quad 3x^2 = \frac{at}{az}$ $\Rightarrow$ $3x^2\, dx = dt$

$\therefore$ From (i), $\quad I = \frac{1}{3} \int t^{1/3}(t+1)\, dt$

$= \frac{1}{3} \int (t^{4/3} + t^{1/3})\, dt$ $\quad \left[\because \frac{1}{3} + 1 = \frac{1+3}{3} = \frac{4}{3}\right]$

$= \frac{1}{3}\left(\int t^{4/3}\, at + \int t^{1/3}\, at\right)$

$$= \frac{1}{3}\left[\frac{t^{7/3}}{\frac{7}{3}} + \frac{t^{4/3}}{\frac{4}{3}}\right] + c = \frac{1}{3}\left[\frac{3}{7}t^{7/3} + \frac{3}{4}t^{4/3}\right] + c = \frac{1}{7}t^{7/3} + \frac{1}{4}t^{4/3} + c$$

Putting $t = x^3 - 1$, $= \frac{1}{7}(x^3 - 1)^{7/3} + \frac{1}{4}(x^3 - 1)^{4/3} + c$.

13. $\dfrac{x^2}{(2 + 3x^3)^3}$

Sol. Let $I = \int \frac{x^2}{(2 + 3x^3)^3}\,dx$

$$= \frac{1}{9}\int \frac{9x^2}{(2 + 3x^3)^3}\,dx \quad \text{...}(i) \qquad \left[\because \frac{d}{dx}(2 + 3x^3) = 9x^2\right]$$

Put $2 + 3x^3 = t$. Therefore $9x^2 = \frac{dt}{dx} \Rightarrow 9x^2\,dx = dt$

$\therefore$ From (i), $I = \frac{1}{9}\int t^{-3}\,dt = \frac{1}{9}\frac{t^{-2}}{-2} + c = \frac{-1}{18t^2} + c$

Putting $t = 2 + 3x^3$; $= \frac{-1}{18(2 + 3x^3)^2} + c$.

14. $\dfrac{1}{x(\log x)^m}$, $x > 0$ ***(Important)***

Sol. Let $I = \int \frac{1}{x(\log x)^m}\,dx \; (x > 0) \Rightarrow I = \int \frac{\frac{1}{x}dx}{(\log x)^m}$ ...(i)

Put $\log x = t$. Therefore $\frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{dx}{x} = dt$

$\therefore$ From (i), $I = \int \frac{dt}{t^m} = \int t^{-m}\,dt = \frac{t^{-m+1}}{-m+1} + c$ (Assuming $m \neq 1$)

Putting $t = \log x$, $= \frac{(\log x)^{1-m}}{1-m} + c$.

15. $\dfrac{x}{9 - 4x^2}$

Sol. Let $I = \int \frac{x}{9 - 4x^2} dx = \frac{-1}{8} \int \frac{-8x}{9 - 4x^2} dx$ ...(*i*)

$\left[\because \frac{d}{dx}(9 - 4x^2) = -8x \right]$

Put $9 - 4x^2 = t$. Therefore $-8x = \frac{dt}{dx} \Rightarrow -8x\,dx = dt$

$\therefore$ From (*i*), $I = \frac{-1}{8} \int \frac{dt}{t} = \frac{-1}{8} \int \frac{1}{t} dt = \frac{-1}{8} \log |t| + c$

Putting $t = 9 - 4x^2$, $= \frac{-1}{8} \log |9 - 4x^2| + c$.

16. e^{2x+3}

Sol. $\int e^{2x+3}\,dx = \dfrac{e^{2x+3}}{2 \to \text{Coeff. of } x} + c$ $\left[\because \int e^{ax+b}\,dx = \dfrac{e^{ax+b}}{a}\right]$

$= \dfrac{1}{2} e^{2x+3} + c.$

17. $\dfrac{x}{e^{x^2}}$

Sol. Let $I = \int \dfrac{x}{(e^{x^2})}\,dx = \dfrac{1}{2}\int \dfrac{2x}{(e^{x^2})}\,dx$ ...(*i*)

Put $x^2 = t$. Therefore $2x = \dfrac{dt}{dx} \Rightarrow 2x\,dx = dt.$

$\therefore$ From (*i*), $I = \dfrac{1}{2}\int \dfrac{dt}{(e^t)} = \dfrac{1}{2}\int e^{-t}\,dt$

$= \dfrac{1}{2}\dfrac{e^{-t}}{-1 \to \text{Coeff. of } t} + c = \dfrac{-1}{2(e^t)} + c$

Putting $t = x^2$, $I = \dfrac{-1}{2(e^{x^2})} + c.$

Integrate the functions in Exercises 18 to 26:

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$

Sol. Let $I = \int \dfrac{e^{\tan^{-1}x}}{1+x^2}\,dx$ ...(*i*)

Put $\tan^{-1}x = t.$

$\therefore \dfrac{1}{1+x^2} = \dfrac{dt}{dx} \Rightarrow \dfrac{dx}{1+x^2} = dt$

$\therefore$ From (*i*), $I = \int e^t\,dt = e^t + c = e^{\tan^{-1}x} + c.$

19. $\dfrac{e^{2x}-1}{e^{2x}+1}$

Sol. Let $I = \int \dfrac{e^{2x}-1}{e^{2x}+1}\,dx$

Multiplying every term in integrand by e^{-x},

$$I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx \qquad ...(i) \quad [\because e^{2x} \cdot e^{-x} = e^{2x-x} = e^x]$$

Put denominator $e^x + e^{-x} = t$

$\therefore \quad e^x + e^{-x} \frac{d}{dx}(-x) = \frac{dt}{dx} \quad \Rightarrow \quad (e^x - e^{-x})\, dx = dt$

$\therefore$ From (i), $I = \int \frac{dt}{t} = \int \frac{1}{t}\, dt = \log|t| + c$

Putting $t = e^x + e^{-x}$, $I = \log|e^x + e^{-x}| + c$ or $I = \log(e^x + e^{-x}) + c$

$\left[\because e^x + e^{-x} = e^x + \frac{1}{(e^x)} > 0 \text{ for all real } x \text{ and hence } |e^x + e^{-x}| = e^x + e^{-x}\right]$

20. $\dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

Sol. Let I = $\displaystyle\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{2(e^{2x} - e^{-2x})}{e^{2x} + e^{-2x}}\,dx$ ...(*i*)

Put denominator $e^{2x} + e^{-2x} = t$

$\therefore\ e^{2x}\dfrac{d}{dx}2x + e^{-2x}\dfrac{d}{dx}(-2x) = \dfrac{dt}{dx}$

$\Rightarrow e^{2x}.2 - 2e^{-2x} = \dfrac{dt}{dx}$ $\Rightarrow 2(e^{2x} - e^{-2x})\,dx = dt$

$\therefore$ From (*i*), $I = \dfrac{1}{2}\displaystyle\int \frac{dt}{t} = \frac{1}{2}\log|t| + c$

Putting $t = e^{2x} + e^{-2x}$, $= \dfrac{1}{2}\log|e^{2x} + e^{-2x}| + c = \dfrac{1}{2}\log(e^{2x} + e^{-2x}) + c$

$[\because e^{2x} + e^{-2x} > 0 \Rightarrow |e^{2x} + e^{-2x}| = e^{2x} + e^{-2x}]$

21. $\tan^2(2x - 3)$

Sol. $\displaystyle\int \tan^2(2x-3)\,dx = \int(\sec^2(2x-3) - 1)\,dx$ $(\because \tan^2\theta = \sec^2\theta - 1)$

$= \displaystyle\int \sec^2(2x-3)\,dx - \int 1\,dx$

$= \dfrac{\tan(2x-3)}{2 \rightarrow \text{Coeff. of } x} - x + c = \dfrac{1}{2}\tan(2x-3) - x + c$

$\left[\because \displaystyle\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c\right]$

22. $\sec^2(7 - 4x)$

Sol. $\displaystyle\int \sec^2(7-4x)\,dx = \frac{\tan(7-4x)}{-4 \rightarrow \text{Coeff. of } x} + c$

$\left[\because \displaystyle\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c\right]$

$= \dfrac{-1}{4}\tan(7-4x) + c.$

23. $\dfrac{\sin^{-1}x}{\sqrt{1 - x^2}}$

Sol. Let $I = \int \frac{\sin^{-1} x \; dx}{\sqrt{1-x^2}}$ ...(*i*)

Put $\sin^{-1} x = t$ $\therefore \frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx}$ $\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

$\therefore$ From (*i*), $I = \int t \; dt = \frac{t^2}{2} + c$

Putting $t = \sin^{-1} x$, $I = \frac{1}{2} (\sin^{-1} x)^2 + c$.

24. $\frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x}$

Sol. Let I = $\int \dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}\, dx = \int \dfrac{2\cos x - 3\sin x}{2(2\sin x + 3\cos x)}\, dx$

$$= \frac{1}{2}\int \frac{2\cos x - 3\sin x}{2\sin x + 3\cos x}\, dx \qquad ...(i)$$

Put DENOMINATOR $2\sin x + 3\cos x = t$

$\therefore\ 2\cos x - 3\sin x = \dfrac{dt}{dx} \Rightarrow (2\cos x - 3\sin x)\, dx = dt$

$\therefore$ From (i), $I = \dfrac{1}{2}\int \dfrac{dt}{t} = \dfrac{1}{2}\log|t| + c.$

Putting $t = 2\sin x + 3\cos x$, $= \dfrac{1}{2}\log|2\sin x + 3\cos x| + c.$

25. $\dfrac{1}{\cos^2 x\,(1 - \tan x)^2}$

Sol. Let I = $\int \dfrac{1}{\cos^2 x\,(1-\tan x)^2}\, dx = \int \dfrac{\sec^2 x}{(1-\tan x)^2}\, dx$

$$= -\int \frac{-\sec^2 x}{(1-\tan x)^2}\, dx \qquad ...(i)$$

Put $1 - \tan x = t$.

$\therefore\ -\sec^2 x = \dfrac{dt}{dx} \Rightarrow -\sec^2 x\, dx = dt$

$\therefore$ From (i), $I = -\int \dfrac{dt}{t^2} = -\int t^{-2}\, dt$

$$= -\frac{t^{-1}}{-1} + c = \frac{1}{t} + c = \frac{1}{1-\tan x} + c.$$

26. $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

Sol. Let I = $\int \dfrac{\cos\sqrt{x}}{\sqrt{x}}\, dx \qquad ...(i)$

Put $\sqrt{\textbf{Linear}} = t$, *i.e.*, $\sqrt{x} = t$

Squaring, $x = t^2$. Therefore $\dfrac{dx}{dt} = 2t \qquad \therefore\ dx = 2t\, dt$

$\therefore$ From (i), I =

$\underline{\cos t\, t}$ $\quad 2t\,dt$

$$= 2\int \cos t \; dt = 2 \sin t + c$$

Putting $t = \sqrt{x}$, $I = 2 \sin \sqrt{x} + c$.

Integrate the functions in Exercises 27 to 37:

27. $\sqrt{\sin 2x}\ \cos 2x$

Sol. Let $I = \int \sqrt{\sin 2x}\ \cos 2x\ dx = \frac{1}{2}\int \sqrt{\sin 2x}\ (2 \cos 2x\ dx)$ ...(i)

Put $\sin 2x = t$

$\therefore \quad \cos 2x \frac{d}{dx}(2x) = \frac{dt}{dx} \quad \Rightarrow \; 2 \cos 2x\ dx = dt$

$\therefore$ From (*i*), $I = \frac{1}{2}\int \sqrt{t}\, dt = \frac{1}{2}\int t^{1/2}\, dt$

$= \frac{1}{2}\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{2}\frac{t^{3/2}}{\frac{3}{2}} + c = \frac{1}{3}(\sin 2x)^{3/2} + c.$

28. $\frac{\cos x}{\sqrt{1+\sin x}}$

Sol. Let $I = \int \frac{\cos x}{\sqrt{1+\sin x}}\, dx$ ...(*i*)

Put $1 + \sin x = t$

$\therefore$ $\cos x = \frac{dt}{dx}$ or $\cos x\, dx = dt$

$\therefore$ From (*i*), $I = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + c$

$= \frac{t^{1/2}}{\frac{1}{2}} + c = 2\sqrt{t} + c = 2\sqrt{1+\sin x} + c.$

29. **$\cot x \log \sin x$**

Sol. Let $I = \int \cot x \log \sin x\, dx$ ...(*i*)

Put $\log \sin x = t$

$\therefore$ $\frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{dt}{dx}$ or $\frac{1}{\sin x}\cos x = \frac{dt}{dx}$

or $\cot x\, dx = dt$

$\therefore$ From (*i*), $I = \int t\, dt = \frac{t^2}{2} + c = \frac{1}{2}(\log \sin x)^2 + c.$

30. $\frac{\sin x}{1+\cos x}$

Sol. Let $I = \int \frac{\sin x}{1+\cos x}\, dx = -\int \frac{-\sin x}{1+\cos x}\, dx$ ...(*i*)

Put $1 + \cos x = t$. Therefore $-\sin x = \frac{dt}{dx}$

$\therefore \quad -\sin x\, dx = dt$

$\therefore \quad$ From (i), $I = -\int \frac{dt}{t} \qquad = -\log|t| + c$

Putting $t = 1 + \cos x$, $= -\log|1 + \cos x| + c$.

31. $\dfrac{\sin x}{(1+\cos x)^2}$

Sol. Let $I = \int \frac{\sin x}{(1+\cos x)^2}\, dx = -\int \frac{-\sin x\, dx}{(1+\cos x)^2}$ ...(i)

Put $1 + \cos x = t$. Therefore $-\sin x = \frac{dt}{dx}$

$\Rightarrow \quad -\sin x\, dx = dt$

$\therefore$ From (i), $I = -\int \frac{dt}{t^2} = -\int t^{-2}\, dt = \frac{-t^{-1}}{-1} + c$

$$= \frac{1}{t} + c = \frac{1}{1+\cos x} + c.$$

32. $\dfrac{1}{1+\cot x}$

Sol. Let $I = \int \frac{1}{1+\cot x} dx = \int \frac{1}{1+\frac{\cos x}{\sin x}} dx = \int \frac{1}{\left(\frac{\sin x + \cos x}{\sin x}\right)} dx$

$$= \int \frac{\sin x}{\sin x + \cos x} dx = \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x} dx = \frac{1}{2}\int \frac{\sin x + \sin x}{\sin x + \cos x} dx$$

Adding and subtracting $\cos x$ in the numerator of integrand,

$$I = \frac{1}{2}\int \frac{\sin x + \cos x - \cos x + \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \left(\frac{\sin x + \cos x}{\sin x + \cos x} - \frac{(\cos x - \sin x)}{\sin x + \cos x}\right) dx \qquad \left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \frac{1}{2}\int \left(1 - \frac{(\cos x - \sin x)}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}\left[\int 1\, dx - \int \frac{\cos x - \sin x}{\sin x + \cos x} dx\right] = \frac{1}{2}[x - I_1] \qquad \ldots(i)$$

where $I_1 = \int \frac{\cos x - \sin x}{\sin x + \cos x} dx$

Put DENOMINATOR $\sin x + \cos x = t$

$\therefore \quad \cos x - \sin x = \frac{dt}{dx} \quad \Rightarrow \quad (\cos x - \sin x)\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log|t| = \log|\sin x + \cos x|.$

Note. Alternative solution for finding I_1

$$I_1 = \int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx = \log|\sin x + \cos x|$$

$$\left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)|\right]$$

Putting this value of I_1 in (i), required integral

$$= \frac{1}{2}[x - \log|\sin x + \cos x|] + c.$$

33. $\dfrac{1}{1-\tan x}$

Sol. Let I $= \int \dfrac{1}{1-\tan x}\,dx = \int \dfrac{1}{1-\dfrac{\sin x}{\cos x}}\,dx = \int \dfrac{1}{\left(\dfrac{\cos x-\sin x}{\cos x}\right)}\,dx$

$= \int \dfrac{\cos x}{\cos x-\sin x}\,dx = \dfrac{1}{2}\int \dfrac{2\cos x}{\cos x-\sin x}\,dx = \dfrac{1}{2}\int \dfrac{\cos x+\cos x}{\cos x-\sin x}\,dx$

Subtracting and adding sin x in the Numerator,

$= \dfrac{1}{2}\int \dfrac{\cos x-\sin x+\sin x+\cos x}{\cos x-\sin x}\,dx$

$= \dfrac{1}{2}\int \left(\dfrac{\cos x-\sin x}{\cos x-\sin x}+\dfrac{\sin x+\cos x}{\cos x-\sin x}\right)dx = \dfrac{1}{2}\int \left(1+\dfrac{\sin x+\cos x}{\cos x-\sin x}\right)dx$

$= \dfrac{1}{2}\left[\int 1\,dx - \int \dfrac{-\sin x-\cos x}{\cos x-\sin x}\,dx\right]$

$= \dfrac{1}{2}[x - \log|\cos x-\sin x|] + c \quad \left[\because \int \dfrac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$

Note. Alternative solution for evaluating $\int \dfrac{-\sin x-\cos x}{\cos x-\sin x}\,dx$, put denominator $\cos x - \sin x = t$.

34. $\dfrac{\sqrt{\tan x}}{\sin x\cos x}$

Sol. Let I $= \int \dfrac{\sqrt{\tan x}}{\sin x\cos x}\,dx = \int \dfrac{\sqrt{\tan x}}{\dfrac{\sin x}{\cos x}\cos x\cos x}\,dx$

$= \int \dfrac{\sqrt{\tan x}}{\tan x\cos^2 x}\,dx = \int \dfrac{\sec^2 x}{\sqrt{\tan x}}\,dx \quad ...(i) \quad \left[\because \dfrac{\sqrt{t}}{t} = \dfrac{1}{\sqrt{t}}\right]$

Put tan x = t.

$\therefore \quad \sec^2 x = \dfrac{dt}{dx} \quad \Rightarrow \quad \sec^2 x\,dx = dt$

$\therefore$ From (i),

$I = \int \dfrac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt = \dfrac{t^{1/2}}{\frac{1}{2}} + c = 2\sqrt{t} + c = 2\sqrt{\tan x} + c.$

35. $\dfrac{(1 + \log x)^2}{x}$

Sol. Let I = $\displaystyle\int \frac{(1+\log x)^2}{x}\,dx$ ...(*i*)

Put $1 + \log x = t$

$\therefore \quad \dfrac{1}{x} = \dfrac{dt}{dx} \quad \Rightarrow \quad \dfrac{dx}{x} = dt$

$\therefore$ From (*i*), I = $\displaystyle\int t^2\, dt = \frac{t^3}{3} + c = \frac{1}{3}(1 + \log x)^3 + c.$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$

Sol. Let I = $\int \dfrac{(x+1)(x+\log x)^2}{x}\, dx$ ...(*i*)

Put $x + \log x = t$

$\therefore\ 1 + \dfrac{1}{x} = \dfrac{dt}{dx} \Rightarrow \dfrac{x+1}{x} = \dfrac{dt}{dx} \Rightarrow \left(\dfrac{x+1}{x}\right) dx = dt$

$\therefore$ From (*i*), I = $\int t^2\, dt = \dfrac{t^3}{3} + c$

Putting $t = x + \log x$, $\dfrac{1}{3}(x + \log x)^3 + c$.

37. $\dfrac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}$

Sol. Let I = $\int \dfrac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}\, dx = \dfrac{1}{4}\int \sin(\tan^{-1} x^4)\, .\, \dfrac{4x^3}{1+x^8}\, dx$ ...(*i*)

Put $(\tan^{-1} x^4) = t$

[Rule for $\int \sin(f(x))\, f'(x)\, dx$; put $f(x) = t$]

$\therefore\ \dfrac{1}{1+(x^4)^2}\dfrac{d}{dx}x^4 = \dfrac{dt}{dx}\ \left[\because \dfrac{d}{dx}\tan^{-1} f(x) = \dfrac{1}{1+(f(x))^2}\dfrac{d}{dx}f(x)\right]$

$\Rightarrow \dfrac{4x^3}{1+x^8}\, dx = dt$

$\therefore$ From (*i*),

I = $\dfrac{1}{4}\int \sin t\, dt = -\dfrac{1}{4}\cos t + c = \dfrac{-1}{4}\cos(\tan^{-1} x^4) + c$.

Choose the correct answer in Exercises 38 and 39:

38. $\int \dfrac{10x^9 + 10^x \log_e 10\, dx}{x^{10} + 10^x}$ **equals**

(A) $10^x - x^{10} + C$ **(B)** $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$ **(D)** $\log(10^x + x^{10}) + C$.

Sol. Let I = $\int \dfrac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\, dx$ ...(*i*)

Put $x^{10} + 10^x = t$

$\therefore \quad (10x^9 + 10^x \log_e 10)\, dx = dt$ $\left[\because \frac{d}{dx}(a^x) = a^x \log_e a\right]$

dt

$\therefore \quad$ From (*i*), $I = \int \frac{\quad}{t} = \log|t| + c$

Putting $t = x^{10} + 10^x$, $I = \log|x^{10} + 10^x| + c$

or $I = \log(10^x + x^{10}) + c$.

$\therefore$ Option (D) is the correct answer.

OR

$$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\, dx = \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c$$

$$= \log\left|x^{10} + 10^x\right| + c$$

$\therefore$ Option (D) is the correct answer.

39. $\int \frac{dx}{\sin^2 x \cos^2 x}$ **equals**

(A) $\tan x + \cot x + C$ **(B)** $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ **(D)** $\tan x - \cot 2x + C$.

Sol. $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx$ $[\because\ 1 = \sin^2 x + \cos^2 x]$

$= \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx$ $\left[\because\ \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right]$

$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx = \int (\sec^2 x + \text{cosec}^2 x)\ dx$

$= \int \sec^2 x\ dx + \int \text{cosec}^2 x\ dx = \tan x - \cot x + c$

$\therefore$ Option (B) is the correct answer.

Exercise 7.3

Find the integrals of the following functions in Exercises 1 to 9:

1. $\sin^2(2x + 5)$

Sol. $\int \sin^2(2x+5)\ dx = \int \frac{1}{2}(1 - \cos 2(2x+5))\ dx$

$\left[\because \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\ ;\ \text{put } \theta = 2x+5\right]$

$= \frac{1}{2}\int (1 - \cos(4x+10))\ dx \quad = \frac{1}{2}\left[\int 1\, dx - \int \cos(4x+10)\, dx\right]$

$= \frac{1}{2}\left[x - \frac{\sin(4x+10)}{4 \to \text{Coeff. of } x}\right] + c = \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + c.$

2.* sin 3*x* cos 4*x

Sol. $\int \sin 3x \cos 4x\ dx = \frac{1}{2}\int 2\sin 3x\cos 4x\ dx$

$= \frac{1}{2}\int (\sin(3x+4x) + \sin(3x-4x))\ dx$

$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$

$= \frac{1}{2}\int (\sin 7x + \sin(-x))\ dx \quad = \frac{1}{2}\int (\sin 7x - \sin x)\ dx$

$= \frac{1}{2}\left[\int \sin 7x\, dx - \int \sin x\, dx\right] = \frac{1}{2}\left[\frac{-\cos 7x}{7} - (-\cos x)\right] + c$

$= \frac{-1}{14}\cos 7x + \frac{1}{2}\cos x + c.$

3.* cos 2*x* cos 4*x* cos 6*x

Sol. $\cos 2x \cos 4x \cos 6x = \frac{1}{2}(2\cos 6x\cos 4x)\cos 2x$

$= \frac{1}{2}[\cos(6x+4x) + \cos(6x-4x)]\cos 2x$

$[\because 2\cos x \,.\, \cos y = \cos(x+y) + \cos(x-y)]$

$= \frac{1}{2}(\cos 10x + \cos 2x)\cos 2x = \frac{1}{4}(2\cos 10x \cos 2x + 2\cos^2 2x)$

$= \frac{1}{4}[\cos(10x + 2x) + \cos(10x - 2x) + 1 + \cos 4x]$

$= \frac{1}{4}(\cos 12x + \cos 8x + \cos 4x + 1)$

$\therefore \int \cos 2x \cos 4x \cos 6x \, dx = \frac{1}{4}\int (\cos 12x + \cos 8x + \cos 4x + 1)\, dx$

$= \frac{1}{4}\left[\int \cos 12x\, dx + \int \cos 8x\, dx + \int \cos 4x\, dx + \int 1\, dx\right]$

$= \frac{1}{4}\left(\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} + x\right) + c.$

Note. We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore\ 4\sin^3\theta = 3\sin\theta - \sin 3\theta$

Dividing by 4, $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$ ...(*i*)

Similarly, $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$ ...(*ii*)

$[\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta]$

4. $\sin^3(2x + 1)$

Sol. To evaluate $\int \sin^3(2x + 1)\, dx$

We know by Eqn. (*i*) of above note that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

Putting $\theta = 2x + 1$, we have

$\sin^3(2x + 1) = \frac{3}{4}\sin(2x + 1) - \frac{1}{4}\sin 3(2x + 1)$

$= \frac{3}{4}\sin(2x + 1) - \frac{1}{4}\sin(6x + 3)$

$\therefore \int \sin^3(2x + 1)\, dx = \frac{3}{4}\int \sin(2x + 1)\, dx - \frac{1}{4}\int \sin(6x + 3)\, dx$

$= \frac{3}{4}\left(\frac{-\cos(2x + 1)}{2}\right) - \frac{1}{4}\left(\frac{-\cos(6x + 3)}{6 \rightarrow \text{Coeff. of } x}\right) + c$

$= \frac{-3}{8}\cos(2x + 1) + \frac{1}{24}\cos(6x + 3) + c.$

OR

To integrate $\sin^n x$ where n is odd, put $\cos x = t$.

$\therefore \int \sin^3(2x + 1)\, dx = \int \sin^2(2x + 1)\sin(2x + 1)\, dx$

$$= \frac{-1}{2} \int [1 - \cos^2 (2x + 1)] \; (-2 \sin (2x + 1)) \, dx \qquad ...(i)$$

Put $\cos (2x + 1) = t$

$\therefore \; -\sin (2x + 1) \frac{d}{dx} (2x + 1) = \frac{dt}{dx} \quad \therefore \; -2 \sin (2x + 1) \, dx = dt$

$\therefore$ From (*i*), the given integral $= \frac{-1}{2} \int (1 - t^2) \, dt$

$$= \frac{-1}{2}\left(t - \frac{t^3}{3}\right) + c = \frac{-1}{2} t + \frac{1}{6} t^3 + c$$

$$= \frac{-1}{2} \cos(2x+1) + \frac{1}{6} \cos^3(2x+1) + c.$$

5. $\sin^3 x \cos^3 x$

Sol. $\int \sin^3 x \cos^3 x \, dx = \int (\sin x \cos x)^3 \, dx$

$$= \int \left(\frac{1}{2} 2 \sin x \cos x\right)^3 dx = \int \left(\frac{1}{2} \sin 2x\right)^3 dx$$

$$= \frac{1}{8} \int \sin^3 2x \, dx = \frac{1}{8} \int \left(\frac{3}{4} \sin 2x - \frac{1}{4} \sin 6x\right) dx$$

$$\left[\text{Putting } \theta = 2x \text{ in } \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right]$$

$$= \frac{3}{32} \int \sin 2x \, dx - \frac{1}{32} \int \sin 6x \, dx$$

$$= \frac{-3}{32} \frac{\cos 2x}{2} - \frac{1}{32}\left(\frac{-\cos 6x}{6}\right) + c = \frac{-3}{64} \cos 2x + \frac{1}{192} \cos 6x + c.$$

OR

To evaluate $\int \sin^3 x \cos^3 x \, dx$, Put either $\sin x = t$ or $\cos x = t$. (The form of answer given in N.C.E.R.T. book II can be obtained by putting $\cos x = t$)

6. **$\sin x \sin 2x \sin 3x$**

Sol. $\sin x \sin 2x \sin 3x = \frac{1}{2} (2 \sin 3x \sin 2x) \sin x$

$$= \frac{1}{2} [\cos(3x - 2x) - \cos(3x + 2x)] \sin x$$

$$[\because 2 \sin x \sin y = \cos(x - y) - \cos(x + y)]$$

$$= \frac{1}{2} (\cos x - \cos 5x) \sin x = \frac{1}{4} (2 \cos x \sin x - 2 \cos 5x \sin x)$$

$$= \frac{1}{4} [\sin 2x - \{\sin(5x + x) - \sin(5x - x)\}]$$

$$[\because 2 \cos x \sin y = \sin(x + y) - \sin(x - y)]$$

$$= \frac{1}{4} (\sin 2x - \sin 6x + \sin 4x)$$

$$\therefore \int \sin x \sin 2x \sin 3x \, dx = \frac{1}{4} \int (\sin 2x + \sin 4x - \sin 6x) \, dx$$

$$= \frac{1}{4} \left[\int \sin 2x \, dx + \int \sin 4x \, dx - \int \sin 6x \, dx\right]$$

$$= \frac{1}{4} \left(-\frac{\cos 2x}{2} - \frac{\cos 4x}{4} + \frac{\cos 6x}{6}\right) + c.$$

4 2 4 6

7. $\sin 4x \sin 8x$

Sol. $\int \sin 4x \sin 8x \; dx = \frac{1}{2} \int 2 \sin 4x \sin 8x \; dx$

$= \frac{1}{2} \int [\cos(4x - 8x) - \cos(4x + 8x)] \; dx$

$[\because 2 \sin A \sin B = \cos(A - B) - \cos(A + B)]$

$$= \frac{1}{2}\int (\cos(-4x) - \cos 12x)\, dx = \frac{1}{2}\int (\cos 4x - \cos 12x)\, dx$$

$[\because \cos(-\theta) = \cos\theta]$

$$= \frac{1}{2}\left[\int \cos 4x\, dx - \int \cos 12x\, dx\right] = \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right] + c.$$

8. $\dfrac{1-\cos x}{1+\cos x}$

Sol. $\displaystyle\int \frac{1-\cos x}{1+\cos x}\, dx = \int \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\, dx = \int \tan^2\frac{x}{2}\, dx$

$\left(\because\ 1-\cos\theta = 2\sin^2\frac{\theta}{2} \text{ and } 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right)$

$$= \int\left(\sec^2\frac{x}{2} - 1\right) dx \qquad (\because\ \tan^2\theta = \sec^2\theta - 1)$$

$$= \int \sec^2\frac{x}{2}\, dx - \int 1\, dx = \frac{\tan\frac{x}{2}}{\frac{1}{2} \to \text{Coeff. of } x} - x + c = 2\tan\frac{x}{2} - x + c.$$

9. $\dfrac{\cos x}{1+\cos x}$

Sol. $\displaystyle\int \frac{\cos x}{1+\cos x}\, dx$

Adding and subtracting 1 in the numerator of integrand,

$$= \int \frac{1+\cos x - 1}{1+\cos x}\, dx = \int\left(\frac{1+\cos x}{1+\cos x} - \frac{1}{1+\cos x}\right) dx \qquad \left(\because\ \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right)$$

$$= \int\left(1 - \frac{1}{2\cos^2\frac{x}{2}}\right) dx = \int 1\, dx - \frac{1}{2}\int \sec^2\frac{x}{2}\, dx$$

$$= x - \frac{1}{2}\,\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c = x - \tan\frac{x}{2} + c.$$

Find the integrals of the functions in Exercises 10 to 18:

10. $\sin^4 x$

$\sin^4 x \qquad (\sin^2 x)^2 \qquad \left(\frac{1-\cos 2x}{\ldots}\right)^2$

Sol. $\int \quad dx = \int \quad dx = \int \left(\quad \frac{}{2} \quad \right) dx$

$$= \int \frac{(1 - \cos 2x)^2}{4}\, dx = \frac{1}{4} \int (1 + \cos^2 2x - 2\cos 2x)\, dx$$

$$= \frac{1}{4} \int \left(1 + \left(\frac{1 + \cos 4x}{2}\right) - 2\cos 2x\right) dx \quad \left[\because \cos^2\theta = \frac{1 + \cos 2\theta}{2}\right]$$

$$= \frac{1}{4}\int\left(\frac{2+1+\cos 4x-4\cos 2x}{2}\right)dx = \frac{1}{8}\int (3+\cos 4x - 4\cos 2x)\,dx$$

$$= \frac{1}{8}\left[3\int 1\,dx + \int \cos 4x\,dx - 4\int \cos 2x\,dx\right]$$

$$= \frac{1}{8}\left[3x + \frac{\sin 4x}{4} - \frac{4\sin 2x}{2}\right] + c = \frac{3}{8}x + \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + c$$

11. **$\cos^4 2x$**

Sol. $\int \cos^4 2x\,dx = \int (\cos^2 2x)^2\,dx$

$$= \int\left(\frac{1+\cos 4x}{2}\right)^2 dx = \int \frac{\frac{1}{4}(1+\cos 4x)^2}{} dx$$

$$= \frac{1}{4}\int (1+\cos^2 4x + 2\cos 4x)\,dx$$

$$= \frac{1}{4}\int\left(1+\frac{1+\cos 8x}{2}+2\cos 4x\right)dx \qquad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \frac{1}{4}\int\left(\frac{2+1+\cos 8x+4\cos 4x}{2}\right)dx = \frac{1}{8}\int (3+\cos 8x + 4\cos 4x)\,dx$$

$$= \frac{1}{8}\left[3\int 1\,dx + \int \cos 8x\,dx + 4\int \cos 4x\,dx\right]$$

$$= \frac{1}{8}\left[3x + \frac{\sin 8x}{8} + \frac{4\sin 4x}{4}\right] + c = \frac{3}{8}x + \frac{1}{64}\sin 8x + \frac{1}{8}\sin 4x + c$$

12. $\dfrac{\sin^2 x}{1+\cos x}$

Sol. $\int \dfrac{\sin^2 x}{1+\cos x}\,dx = \int \dfrac{1-\cos^2 x}{1+\cos x}\,dx = \int \dfrac{(1-\cos x)(1+\cos x)}{1+\cos x}\,dx$

$$= \int (1-\cos x)\,dx = \int 1\,dx - \int \cos x\,dx = x - \sin x + c.$$

Note. It may be noted that letters $a, b, c, d, \ldots, q$ of English Alphabet and letters $\alpha, \beta, \gamma, \delta$ of Greek Alphabet are generally treated as constants.

13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

Sol. $\int \dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx = \int \dfrac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}\,dx$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2\alpha + 1}{\cos x - \cos\alpha}\, dx \qquad = \int \frac{2\cos^2 x - 2\cos^2\alpha}{\cos x - \cos\alpha}\, dx$$

$$= 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}\, dx \qquad = 2\int \frac{(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{(\cos x - \cos\alpha)}\, dx$$

$$= 2\int (\cos x + \cos\alpha)\, dx \qquad = 2\left[\int \cos x\, dx + \int \cos\alpha\, dx\right]$$

$$= 2\left[\sin x + \cos\alpha \int 1\, dx\right] = 2\left[\sin x + (\cos\alpha)\, x\right] + c$$
$$= 2\sin x + 2x\cos\alpha + c.$$

Remark. $\int \sin a \; dx = \sin a \int 1 \; dx = x \sin a$.

Please note that $\int \sin a \, dx \neq -\cos a$.

14. $\dfrac{\cos x - \sin x}{1 + \sin 2x}$

Sol. Let $I = \int \dfrac{\cos x - \sin x}{1 + \sin 2x} dx = \int \dfrac{\cos x - \sin x}{\cos^2 x + \sin^2 x + 2 \sin x \cos x} dx$

$$= \int \frac{\cos x - \sin x}{(\cos x + \sin x)^2} \; dx \qquad ...(i)$$

Put $\cos x + \sin x = t$.

$\therefore -\sin x + \cos x = \dfrac{dt}{dx}$. Therefore $(\cos x - \sin x) \, dx = dt$.

$\therefore$ From (*i*), $I = \int \dfrac{dt}{t^2} = \int t^{-2} \; dt = \dfrac{t^{-1}}{-1} + c$

$\Rightarrow \; I = \dfrac{-1}{t} + c = \dfrac{-1}{\cos x + \sin x} + c.$

15. **$\tan^3 2x \sec 2x$**

Sol. Let $I = \int \tan^3 2x \; \sec 2x \; dx = \int \tan^2 2x \tan 2x \sec 2x \; dx$

$$= \int (\sec^2 2x - 1) \sec 2x \tan 2x \; dx \qquad [\because \; \tan^2 \theta = \sec^2 \theta - 1]$$

$$= \frac{1}{2} \int (\sec^2 2x - 1)(2 \sec 2x \; \tan 2x) \; dx \qquad ...(i)$$

Put $\sec 2x = t$. Therefore $\sec 2x \tan 2x \dfrac{d}{dx}(2x) = \dfrac{dt}{dx}$

$\therefore \; 2 \sec 2x \tan 2x \; dx = dt$

$\therefore$ From (*i*), $I = \dfrac{1}{2} \int (t^2 - 1) \; dt = \dfrac{1}{2} \left(\int t^2 \; dt - \int 1 \; dt \right)$

$$= \frac{1}{2} \left(\frac{t^3}{3} - t \right) + c = \frac{1}{6} t^3 - \frac{1}{2} t + c$$

Putting $t = \sec 2x$, $= \dfrac{1}{6} \sec^3 2x - \dfrac{1}{2} \sec 2x + c$.

16. **$\tan^4 x$**

Sol. $\int \tan^4 x \, dx = \int \tan^2 x \tan^2 x \, dx = \int \tan^2 x (\sec^2 x - 1) \, dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x) \, dx = \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$

$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$

$= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int 1 \, dx$
$\downarrow$

For this integral, put **tan *x* = *t*.**

$\therefore \quad \sec^2 x = \frac{dt}{dx} \quad$ or $\quad \sec^2 x \, dx = dt$

$$= \int t^2 \, dt - \tan x + x + c = \frac{t^3}{3} - \tan x + x + c$$

Put $t = \tan x$, $= \frac{1}{3} \tan^3 x - \tan x + x + c.$

17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Sol. $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} \right) dx$

$$\left(\because \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \right)$$

$$= \int \left(\frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \right) dx = \int \left(\frac{\sin x}{\cos x \cos x} + \frac{\cos x}{\sin x \sin x} \right) dx$$

$$= \int (\tan x \sec x + \cot x \operatorname{cosec} x) \, dx$$

$$= \int \sec x \tan x \, dx + \int \operatorname{cosec} x \cot x \, dx = \sec x - \operatorname{cosec} x + c.$$

18. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

Sol. $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \frac{(1 - 2\sin^2 x) + 2\sin^2 x}{\cos^2 x} dx$

$$= \int \frac{1}{\cos^2 x} dx = \int \sec^2 x \, dx = \tan x + c.$$

Integrate the functions in Exercises 19 to 22:

Note. Method to evaluate $\int \frac{1}{\sin^p x \cos^q x} dx$ if $(p + q)$ is a negative even integer $(= -n$ (say)); then multiply Numerator and Denominator of integrand by $\sec^n x$.

19. $\dfrac{1}{\sin x \cos^3 x}$

Sol. Let $I = \int \frac{1}{\sin x \cos^3 x} dx$ ...(*i*)

Here $p + q = -1 - 3 = -4$ is a negative even integer.

So multiplying both Numerator and Denominator of integrand of (i) by $\sec^4 x$,

$$I = \int \frac{\sec^4 x}{\sin x \cos^3 x \sec^4 x}\, dx = \int \frac{\sec^4 x}{\tan x}\, dx$$

$$\left(\because \sin x \cos^3 x \sec^4 x = \sin x \cos^3 x \cdot \frac{1}{\cos^4 x} = \frac{\sin x}{\cos x} = \tan x\right)$$

$$\text{or} \quad I = \int \frac{\sec^2 x \sec^2 x}{\tan x}\, dx = \int \frac{(1 + \tan^2 x)\sec^2 x}{\tan x}\, dx \qquad ...(ii)$$

Put $\tan x = t$

$\therefore \quad \sec^2 x = \frac{dt}{dx} \quad \Rightarrow \quad \sec^2 x\, dx = dt$

$\therefore$ From (*ii*), $I = \int \frac{(1+t^2)}{t} dt = \int \left(\frac{1}{t} + \frac{t^2}{t}\right) dt$

$= \int \left(\frac{1}{t} + t\right) dt = \int \frac{1}{t} dt + \int t\, dt = \log |t| + \frac{t^2}{2} + c$

Putting $t = \tan x$, $= \log |\tan x| + \frac{1}{2}\tan^2 x + c.$

20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$

Sol. Let $I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$

$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)(\cos x + \sin x)} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$ ...(*i*)

Put DENOMINATOR $\cos x + \sin x = t$

$\therefore \quad -\sin x + \cos x = \frac{dt}{dx} \Rightarrow (\cos x - \sin x)\, dx = dt$

$\therefore$ From (*i*), $I = \int \frac{dt}{t} = \log |t| + c = \log |\cos x + \sin x| + c$

Note. Another method to evaluate integral (*i*) is, apply

$\int \frac{f'(x)}{f(x)} dx = \log |f(x)|.$

21. $\sin^{-1}(\cos x)$

Sol. $\int \sin^{-1}(\cos x)\, dx = \int \sin^{-1} \sin\left(\frac{\pi}{2} - x\right) dx$

$= \int \left(\frac{\pi}{2} - x\right) dx = \int \frac{\pi}{2} dx - \int x\, dx$

$= \frac{\pi}{2}\int 1\, dx - \int x^1\, dx = \frac{\pi}{2} x - \frac{x^2}{2} + c.$

22. $\dfrac{1}{\cos (x-a) \cos (x-b)}$

Sol. Let $I = \int \frac{1}{\cos (x-a) \cos (x-b)} dx$ ...(*i*)

Here $(x - a) - (x - b) = x - a - x + b = b - a$ ...(*ii*)

By looking at Eqn. (*ii*), dividing and multiplying the integrand in (i) by sin $(b - a)$,

$$I = \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin[(x-a)-(x-b)]}{\cos(x-a)\cos(x-b)}\, dx \qquad [\text{By } (ii)]$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)}\, dx$$

$[\because \sin(A - B) = \sin A \cos B - \cos A \sin B]$

$$= \frac{1}{\sin(b-a)} \int \left[\frac{\sin(x-a)\cos(x-b)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)} \right] dx$$

$$\left(\because \frac{A-B}{C} = \frac{A}{C} - \frac{B}{C} \right)$$

$$= \frac{1}{\sin(b-a)} \int [\tan(x-a) - \tan(x-b)]\, dx$$

$$= \frac{1}{\sin(b-a)} [-\log|\cos(x-a)| + \log|\cos(x-b)|] + c$$

$$\left(\because \int \tan x \, dx = -\log|\cos x| \right)$$

$$= \frac{1}{\sin(b-a)} \log\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + c. \quad \left(\because \log m - \log n = \log \frac{m}{n} \right)$$

Choose the correct answer in Exercises 23 and 24:

23. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ **is equal to**

(A) $\tan x + \cot x + C$ **(B) $\tan x + \operatorname{cosec} x + C$**

(C) $-\tan x + \cot x + C$ **(D) $\tan x + \sec x + C$**

Sol. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$

$$= \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx \quad \left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c} \right]$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} \right) dx = \int (\sec^2 x - \operatorname{cosec}^2 x)\, dx$$

$$= \int \sec^2 x \, dx - \int \operatorname{cosec}^2 x \, dx = \tan x - (-\cot x) + C$$

$= \tan x + \cot x + C$ $\therefore$ Option (A) is the correct answer.

24. $\int \frac{e^x (1 + x)}{\cos^2 (e^x x)} dx$ **equals**

(A) $-\cot(ex^x) + C$ **(B) $\tan(xe^x) + C$**

(C) $\tan(e^x) + C$ **(D) $\cot(e^x) + C$**

Sol. Let I $= \int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ ...(*i*)

Put $e^x \cdot x = t$

[To evaluate $\int$ (T-function or Inverse T-function $f(x)$) $f'(x)\, dx$, put

$f(x) = t]$

Applying Product Rule, $e^x \cdot 1 + xe^x = \frac{dt}{dx}$

or $e^x (1 + x)\, dx = dt$

$\therefore$ From (i), $I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t \; dt$

$= \tan t + C = \tan (x\, e^x) + C$ $\therefore$ Option (B) is the correct answer.

Exercise 7.4

Integrate the following functions in Exercises 1 to 9:

1. $\dfrac{3x^2}{x^6+1}$

Sol. Let I $= \displaystyle\int \frac{3x^2}{x^6+1}\,dx = \int \frac{3x^2}{(x^3)^2+1^2}\,dx$ ...(i)

Put $x^3 = t$

$\therefore\ 3x^2 = \dfrac{dt}{dx} \Rightarrow 3x^2\,dx = dt$

$\therefore$ From (i), I $= \displaystyle\int \frac{dt}{t^2+1^2} = \frac{1}{1}\tan^{-1}\frac{t}{1} + C$ $\left[\because \displaystyle\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

Putting $t = x^3$; $= \tan^{-1}(x^3) + C$.

Note. $ax^2 + b$ ($a \neq 0$) is called a **pure quadratic.**

2. $\dfrac{1}{\sqrt{1+4x^2}}$

Sol. Let I $= \displaystyle\int \frac{1}{\sqrt{1+4x^2}}\,dx = \int \frac{1}{\sqrt{(2x)^2+1^2}}\,dx$

Using $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x + \sqrt{x^2+a^2}\right|$,

I $= \dfrac{\log\left|(2x)+\sqrt{(2x)^2+1^2}\right|}{2 \to \text{Coeff. of } x} + C = \dfrac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C.$

3. $\dfrac{1}{\sqrt{(2-x)^2+1}}$

Sol. Let I $= \displaystyle\int \frac{1}{\sqrt{(2-x)^2+1}}\,dx = \int \frac{1}{\sqrt{(2-x)^2+1}}\,dx$

Using $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x + \sqrt{x^2+a^2}\right|$,

$= \dfrac{\log\left|(2-x)+\sqrt{(2-x)^2+1^2}\right|}{}$

$$= -\log = 1 \rightarrow \text{Coeff. of } x \quad + C$$

$$\left| 2 - x + \sqrt{4 + x^2 - 4x + 1} \right| + C$$

$$= \log \left| \frac{1}{2 - x + \sqrt{x^2 - 4x + 5}} \right| + C.$$

$$\left[\because \; -\log \frac{m}{n} = -(\log m - \log n) = \log n - \log m = \log \frac{n}{m} \right]$$

4. $\dfrac{1}{\sqrt{9-25x^2}}$

Sol. Let $I = \int \dfrac{1}{\sqrt{9-25x^2}}\,dx = \int \dfrac{1}{\sqrt{3^2-(5x)^2}}\,dx$

$$= \frac{\sin^{-1}\frac{5x}{3}}{5 \rightarrow \text{Coeff. of } x} + C \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C.$$

5. $\dfrac{3x}{1+2x^4}$

Sol. Let $I = \int \dfrac{3x}{1+2x^4}\,dx = \dfrac{3}{2}\int \dfrac{2x}{1+2(x^2)^2}\,dx$ ...(*i*)

Put $x^2 = t$. $\therefore\ 2x = \dfrac{dt}{dx} \Rightarrow 2x\,dx = dt$

$\therefore$ From (*i*), $I = \dfrac{3}{2}\int \dfrac{dt}{1+2t^2} = \dfrac{3}{2}\int \dfrac{1}{(\sqrt{2}t)^2+1^2}\,dt$

$$= \frac{3}{2}\,\frac{\frac{1}{1}\tan^{-1}\frac{\sqrt{2}t}{1}}{\sqrt{2} \rightarrow \text{Coeff. of } t} + C = \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,t) + C$$

Putting $t = x^2$, $= \dfrac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,x^2) + C.$

6. $\dfrac{x^2}{1-x^6}$

Sol. Let $I = \int \dfrac{x^2}{1-x^6}\,dx = \int \dfrac{x^2}{1-(x^3)^2}\,dx = \dfrac{1}{3}\int \dfrac{3x^2}{1-(x^3)^2}\,dx$

Put $x^3 = t$. Therefore $3x^2 = \dfrac{dt}{dx} \Rightarrow 3x^2\,dx = dt.$

$\therefore\ I = \dfrac{1}{3}\int \dfrac{dt}{1-t^2} = \dfrac{1}{3}\int \dfrac{1}{1^2-t^2}\,dt = \dfrac{1}{3}\,\dfrac{1}{2\times 1}\log\left|\dfrac{1+t}{1-t}\right| + C$

$$\left[\because \int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right]$$

Putting $t = x^3$, $= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C.$

7. $\dfrac{x-1}{\sqrt{x^2-1}}$

Sol. Let $I = \int \frac{x-1}{\sqrt{x^2-1}}\,dx = \int \left(\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right) dx$

$$= \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1^2}}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\,dx - \log\left|x+\sqrt{x^2-1^2}\right| \quad ...(i)$$

$$\left(\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right|\right)$$

Let $I_1 = \int \frac{2x}{\sqrt{x^2-1}}\,dx$

Put $x^2 - 1 = t$. Therefore $2x = \frac{dt}{dx} \Rightarrow 2x\,dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{x^2-1} + C$$

Putting this value of $I_1 = \int \frac{2x}{\sqrt{x^2-1}}\,dx$ in (i),

$$I = \frac{1}{2}\left(2\sqrt{x^2-1} + C\right) - \log\left|x+\sqrt{x^2-1}\right|$$

$$= \sqrt{x^2-1} + \frac{C}{2} - \log\left|x+\sqrt{x^2-1}\right|$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C_1 \text{ where } C_1 = \frac{C}{2}.$$

8. $\frac{x^2}{\sqrt{x^6+a^6}}$

Sol. Let $I = \int \frac{x^2}{\sqrt{x^6+a^6}}\,dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{(x^3)^2+a^6}}\,dx \quad ...(i)$

Put $x^3 = t$. Therefore $3x^2 = \frac{dt}{dx} \Rightarrow 3x^2\,dx = dt$.

$$\therefore \quad \text{From } (i),\ I = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+a^6}} = \frac{1}{3}\int \frac{1}{\sqrt{t^2+(a^3)^2}}\,dt$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+(a^3)^2}\right| + C \quad \left[\because \int\right.$$

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$\frac{1}{\quad}$ ⌉ $dx = \log \quad x$ $\sqrt{x^2 + a^2}$ | $\sqrt{x^2 + a^2}$ ||

+

|⌋

Putting $t = x^3$, $= \frac{1}{3} \log \left| x^3 + \sqrt{x^6 + a^6} \right| + C.$

9. $\dfrac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$

Sol. Let $I = \int \dfrac{\sec^2 x}{\sqrt{\tan^2 x + 4}}\, dx$ ...(*i*)

Put $\tan x = t$. $\therefore \sec^2 x = \frac{dt}{dx} \Rightarrow \sec^2 x\, dx = dt$

$\therefore$ From (*i*), $I = \int \frac{dt}{\sqrt{t^2+4}} = \int \frac{1}{\sqrt{t^2+2^2}} dt$

$= \log \left| t + \sqrt{t^2+2^2} \right| + C \quad \left[\because \int \frac{1}{\sqrt{x^2+a^2}} dx = \log \left| x + \sqrt{x^2+a^2} \right| \right]$

Putting $t = \tan x$, $I = \log \left| \tan x + \sqrt{\tan^2 x + 4} \right| + C$.

Integrate the following functions in Exercises 10 to 18:

Note. Rule to evaluate

$$\int \frac{1}{\text{Quadratic}} dx \text{ or } \int \frac{1}{\sqrt{\text{Quadratic}}} dx \text{ or } \int \sqrt{\text{Quadratic}}\, dx$$

Write Quadratic. Take coefficient of x^2 common to make it unity. Then complete squares by adding and subtracting $\left(\frac{1}{2} \text{coefficient of } x\right)^2$

10. $\frac{1}{\sqrt{x^2+2x+2}}$

Sol. $\int \frac{1}{\sqrt{x^2+2x+2}} dx = \int \frac{1}{\sqrt{x^2+2x+1+1}} dx = \int \frac{1}{\sqrt{(x+1)^2+1^2}} dx$

Using $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log \left| x + \sqrt{x^2+a^2} \right|$

$= \log \left| x+1+\sqrt{(x+1)^2+1^2} \right| + c = \log \left| x+1+\sqrt{x^2+2x+2} \right| + c.$

11. $\frac{1}{9x^2+6x+5}$

Sol. Let $I = \int \frac{1}{9x^2+6x+5} dx$ ...(*i*)

$\int \frac{1}{\text{Quadratic}} dx$

Here Quadratic expression $= 9x^2 + 6x + 5$

Making coefficient of x^2 unity, $= 9\left(x^2 + \frac{6x}{9} + \frac{5}{9}\right)$

$9 \quad 9$

$$= 9\left(x^2 + \frac{2x}{3} + \frac{5}{9}\right)$$

To complete squares, adding and subtracting $\left(\frac{1}{2}\text{ Coefficient of } x\right)^2$

$$= \left(\left(\frac{1}{2} \times \frac{2}{3}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}\right) = 9\left(x^2 + \frac{2x}{3} + \left(\frac{1}{3}\right)^2 - \frac{1}{9} + \frac{5}{9}\right)$$

$$= 9\left[\left(x+\frac{1}{3}\right)^2+\frac{4}{9}\right] \Rightarrow 9x^2+6x+5 = 9\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right]$$

Putting this value in (*i*), $\quad I = \int \frac{1}{9\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right]}dx$

$$= \frac{1}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2}dx$$

$$= \frac{1}{9}\cdot\frac{1}{\left(\frac{2}{3}\right)}\tan^{-1}\frac{x+\frac{1}{3}}{\frac{2}{3}}+c \qquad \left(\because \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$= \frac{1}{9}\cdot\frac{3}{2}\tan^{-1}\left(\frac{\frac{3x+1}{3}}{\frac{2}{3}}\right)+c = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+c.$$

12. $\frac{1}{\sqrt{7-6x-x^2}}$

Sol. Let $I = \int \frac{1}{\sqrt{7-6x-x^2}}dx$ ...(*i*) | Type $\int \frac{1}{\text{Quadratic}}dx$

Here Quadratic expression is $7-6x-x^2 = -x^2-6x+7$.

Making coefficient of x^2 unity, $= -(x^2+6x-7)$.

To complete squares, adding and subtracting $\left(\frac{1}{2}\text{ coefficient of } x\right)^2$

$$= \left(\frac{1}{2}\times 6\right)^2 = 9$$

$$= -[x^2+6x+9-9-7] = -[(x+3)^2-16] \qquad ...(ii)$$

$$= -(x+3)^2+16 = 4^2-(x+3)^2 \qquad ...(iii)$$

(**Note.** Must adjust negative sign outside Eqn. (*ii*) in the bracket as shown above because otherwise we shall get $\sqrt{-1} = i$ on taking square roots.]

Putting the value of quadratic expression from (*iii*) in (*i*),

$$I = \int \frac{1}{\sqrt{4^2-(x+3)^2}}dx = \sin^{-1}\left(x+3\right) + c$$

13. $\frac{1}{\sqrt{(x-1)(x-2)}}$

$\int \frac{1}{\sqrt{(x-1)(x-2)}}$ ... $\sqrt{x^2-3x+2}$

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$\left[\because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}\right]$

Sol. Let I = $\int$ $dx = \int \frac{1}{\quad} dx$

$$= \int \frac{1}{\sqrt{x^2 - 3x + 2}} \quad ...(i)$$

Here quadratic expression is $x^2 - 3x + 2$. Coefficient of x^2 is already unity. To complete squares, adding and subtracting $\left(\frac{1}{2}\text{ coefficient of } x\right)^2$, *i.e.,* $\left(-\frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$

$$x^2 - 3x + 2 = x^2 - 3x + \left(\frac{3}{2}\right)^2 - \frac{9}{4} + 2$$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \qquad \left[\because \frac{-9}{4} + 2 = \frac{-9+8}{4} = \frac{-1}{4}\right]$$

$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \quad ...(ii)$$

Putting this value in (*i*), I = $\int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$

$$= \log \left| x - \frac{3}{2} + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log \left| x + \sqrt{x^2 - a^2} \right|\right]$$

$$= \log \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + 2} \right| + c. \quad \text{[By (ii)]}$$

14. $\frac{1}{\sqrt{8 + 3x - x^2}}$

Sol. Let I = $\int \frac{1}{\sqrt{8 + 3x - x^2}} dx$ ...(*i*)

Here quadratic expression is $8 + 3x - x^2 = -x^2 + 3x + 8$.
Making coefficient of x^2 unity, $= -(x^2 - 3x - 8)$.
To complete squares, adding and subtracting $\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{3}{2}\right)^2$

$$8 + 3x - x^2 = -\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - \right.$$

$$= -\left[\left(x-\frac{3}{2}\right)^2 - \frac{9}{4} - 8\right] = -\left[\left(x-\frac{3}{2}\right)^2 - \frac{41}{4}\right] = \frac{41}{4} - \left(x-\frac{3}{2}\right)^2$$

(See **Note** given in the solution of Q.N. 12)

$$= \left(\frac{\sqrt{41}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 \qquad \ldots(ii)$$

Putting this value in (i), $I = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} dx$

$= \sin^{-1} \frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}} + c$ $\left[\because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}\right]$

$= \sin^{-1} \left(\frac{2x - 3}{\sqrt{41}}\right) + c.$

15. $\frac{1}{\sqrt{(x-a)(x-b)}}$

Sol. Let $I = \int \frac{1}{\sqrt{(x-a)(x-b)}} dx = \int \frac{1}{\sqrt{x^2 - bx - ax + ab}} dx$

$= \int \frac{1}{\sqrt{x^2 - x(a+b) + ab}}$ ...(i)

Here Quadratic expression $= x^2 - x(a+b) + ab$

Adding and subtracting $\left(\frac{1}{2} \text{coefficient of } x\right)^2 = \left(\frac{a+b}{2}\right)^2$

$= x^2 - x(a+b) + \left(\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab$

$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{(a+b)^2}{4} - ab\right)$

$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{(a+b)^2 - 4ab}{4}\right) = \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \frac{(a-b)^2}{4}$

$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2$ ...(ii)

$(\because (a+b)^2 - 4ab = a^2 + b^2 + 2ab - 4ab = a^2 + b^2 - 2ab = (a-b)^2)$

Putting this value in (i),

$I = \int \frac{1}{\sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2}} dx$

$$= \log\left|x-\left(\frac{a+b}{2}\right)+\sqrt{\left(x-\left(\frac{a+b}{2}\right)\right)^2-\left(\frac{a-b}{2}\right)^2}\right|+c$$

$$\left[\because \int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \log \left| x - \left(\frac{a+b}{2}\right) + \sqrt{x^2 - x(a+b) + ab} \right| + c \quad \text{[By } (ii)\text{]}$$

Note. Method to evaluate $\int \frac{\textbf{Linear}}{\textbf{Quadratic}} dx$ **or** $\int \frac{\textbf{Linear}}{\sqrt{\textbf{Quadratic}}} dx$

or $\int \textbf{Linear} \sqrt{\textbf{Quadratic}}\ dx$.

Write linear = A $\frac{d}{dx}$ (Quadratic) + B.

Find values of A and B by comparing coefficients of x and constant terms on both sides.

16. $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Sol. Let I = $\int \frac{4x+1}{\sqrt{2x^2+x-3}} dx$ ...(*i*)

Here $\frac{d}{dx}$ (Quadratic $2x^2 + x - 3$) is $(4x + 1)$, the numerator.

So put $2x^2 + x - 3 = t$.

$\therefore$ $(4x + 1) = \frac{dt}{dx}$ $\Rightarrow$ $(4x + 1)\, dx = dt$

$\therefore$ From (*i*), I = $\int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} + c$

$= 2\sqrt{t} + c = 2\sqrt{2x^2 + x - 3} + c.$

17. $\frac{x+2}{\sqrt{x^2-1}}$

Sol. Let I = $\int \frac{x+2}{\sqrt{x^2-1}} dx = \int \left(\frac{x}{\sqrt{x^2-1}} + \frac{2}{\sqrt{x^2-1}} \right) dx$

$= \int \frac{x}{\sqrt{x^2-1}} dx + 2\int \frac{1}{\sqrt{x^2-1^2}} dx$

$= \int \frac{x}{\sqrt{x^2-1}} dx + 2 \log \left| x + \qquad \right| + c$ ...(*i*)

Let I_1 =

$$\int \frac{x}{\sqrt{x^2 - 1}}\, dx = \frac{1}{2} \int \frac{2x}{\sqrt{x^2 - 1}}\, dx$$

Put $x^2 - 1 = t$. Therefore $2x = \frac{dt}{dx}$ or $2x\, dx = dt$

$$\therefore\ I = \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \int t^{-1/2}\, dt = \frac{1}{2} \frac{t^{1/2}}{\frac{1}{2}} = \sqrt{t} = \sqrt{x^2 - 1}$$

Putting this value of (I_1 =) $\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}$ in (*i*)

$$I = \sqrt{x^2-1} + 2\log\left|x + \sqrt{x^2-1}\right| + c.$$

18. $\dfrac{5x-2}{1+2x+3x^2}$

Sol. Let $I = \int \frac{5x-2}{1+2x+3x^2}\,dx$ ...(*i*) $\left| \int \frac{\text{Linear}}{\text{Quadratic}}\,dx \right.$

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $5x - 2 = A\frac{d}{dx}(1 + 2x + 3x^2) + B$

or $5x - 2 = A(2 + 6x) + B$ ...(*ii*)

i.e., $5x - 2 = 2A + 6Ax + B$

Comparing coefficients of x, $6A = 5 \Rightarrow A = \frac{5}{6}$

Comparing constants, $2A + B = -2$

Putting $A = \frac{5}{6}$, $\frac{10}{6} + B = -2$

$\Rightarrow B = -2 - \frac{10}{6} = \frac{-22}{6}$ or $B = \frac{-11}{3}$

Putting values of A and B in (*ii*), $5x - 2 = \frac{5}{6}(2 + 6x) - \frac{11}{3}$

Putting this value of $5x - 2$ in (*i*),

$$I = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}\,dx$$

$$\Rightarrow I = \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}\,dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}\,dx$$

$$= \frac{5}{6} I_1 - \frac{11}{3} I_2 \quad ...(iii)$$

Here $I_1 = \int \frac{2+6x}{1+2x+3x^2}\,dx$

Put Denominator $1 + 2x + 3x^2 = t.$

dt

$\therefore \; 2 + 6x = \frac{dt}{dx} \quad \Rightarrow \quad (2 + 6x)\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \int \frac{1}{t}\, dt = \log|t| = \log|1 + 2x + 3x^2| \quad \ldots(iv)$

Again $I_2 = \int \frac{1}{1 + 2x + 3x^2}\, dx = \int \frac{1}{3x^2 + 2x + 1}\, dx \,\Big|\, \int \frac{1}{\text{Quadratic}}\, dx$

Now Quadratic Expression $= 3x^2 + 2x + 1$.

Making coefficient of x^2 unity $= 3\left(x^2 + \frac{2}{3}x + \frac{1}{3}\right)$

Completing squares $= 3\left[x^2+\frac{2}{3}x+\left(\frac{1}{3}\right)^2+\frac{1}{3}-\frac{1}{9}\right]$

$= 3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$ $\quad \because \frac{1}{3}-\frac{1}{9}=\frac{3-1}{9}=\frac{2}{9}$

$\Rightarrow \quad I_2 = \int \frac{1}{3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]}dx = \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}dx$

$= \frac{1}{3}\cdot\frac{1}{\left(\frac{\sqrt{2}}{3}\right)}\tan^{-1}\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}$ $\quad \left[\because \int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$\Rightarrow \quad I_2 = \frac{1}{3}\cdot\frac{3}{\sqrt{2}}\tan^{-1}\frac{3x+1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)$ ...(*v*)

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*), we have

$I = \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+c.$

Integrate the functions in Exercises 19 to 23:

19. $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

Sol. Let $I = \int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = \int \frac{6x+7}{\sqrt{x^2-4x-5x+20}}dx$

i.e., $I = \int \frac{6x+7}{\sqrt{x^2-9x+20}}dx$ ...(*i*) $\quad \Big| \int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}}dx$

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $6x+7 = A(2x-9)+B$ ...(*ii*)

$= 2Ax - 9A + B$

Comparing coefficients of x, $2A = 6 \Rightarrow A = 3$

Comparing constants, $-9A + B = 7$.

Putting $A = 3$, $-27 + B = 7 \Rightarrow B = 34$

Putting values of A and B in (*ii*),

$6x + 7 = 3(2x - 9) + 34$

Putting this value of $6x + 7$ in (*i*),

$$I = \int \frac{3(2x-9)+34}{x^2-9x+20}\,dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}\,dx + 34 \int \qquad dx$$

$$= 3\,I_1 + 34\,I_2 \qquad \ldots(iii)$$

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}\,dx$$

Put $x^2 - 9x + 20 = t$. $\therefore$ $2x - 9 = \frac{dt}{dx}$

$\Rightarrow \quad (2x - 9)\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{1/2} = 2\sqrt{t}$

$= 2\sqrt{x^2 - 9x + 20}$ ...(*iv*)

$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}}\, dx = \int \frac{1}{\sqrt{x^2 - 9x + \left(\frac{9}{2}\right)^2 + 20 - \frac{81}{4}}}\, dx$

$= \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}}\, dx = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$

$= \log \left| x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right|$

$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log|x + \sqrt{x^2 - a^2}| \right]$

$I_2 = \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right|$ ...(*v*)

$\left(\because \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = x^2 + \frac{81}{4} - 9x - \frac{1}{4} = x^2 - 9x + 20 \right)$

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*),

$I = 6\sqrt{x^2 - 9x + 20} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + c.$

20. $\frac{x + 2}{\sqrt{4x - x^2}}$

Sol. Let $I = \int \frac{x + 2}{\sqrt{4x - x^2}}\, dx$ ...(*i*) $\left| \int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}}\, dx \right.$

Let Linear $= A \frac{d}{dx}$ (Quadratic) $+ B$

i.e., $x + 2 = A(4 - 2x) + B$ ...(*ii*)

$= 4A - 2Ax + B$

Comparing coefficients of x: $-2A = 1 \Rightarrow A = \frac{-1}{2}$

Comparing constants: $4A + B = 2$

Putting $A = \frac{-1}{}$, $-2 + B = 2 \Rightarrow B = 4$

Putting values of A and B in (*ii*), $x + 2 = \frac{-1}{2}(4 - 2x) + 4$

Putting this value of $x + 2$ in (*i*),

$$I = \int \frac{\frac{-1}{2}(4-2x)+4}{\sqrt{4x-x^2}}\,dx = \frac{-1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx + 4\int \frac{1}{\sqrt{4x-x^2}}\,dx$$

$$= \frac{-1}{2} I_1 + 4 I_2 \quad ...(iii) \qquad I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}\,dx$$

Put $4x - x^2 = t \quad \therefore \quad 4 - 2x = \frac{dt}{dx} \quad \Rightarrow \quad (4-2x)\,dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt = \frac{t^{1/2}}{1/2} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad ...(iv)$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}}\,dx$$

Quadratic Expression is $4x - x^2 = -x^2 + 4x$
$= -(x^2 - 4x) = -(x^2 - 4x + 4 - 4) = -((x-2)^2 - 2^2) = 2^2 - (x-2)^2$

$$\therefore \quad I_2 = \int \frac{1}{\sqrt{2^2-(x-2)^2}}\,dx = \sin^{-1}\frac{x-2}{2} \qquad ...(v)$$

$$\left(\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}\right)$$

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*),

$$I = -\sqrt{4x-x^2} + 4\sin^{-1}\frac{x-2}{2} + c.$$

21. $\dfrac{x+2}{\sqrt{x^2+2x+3}}$

Sol. Let $I = \int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx \qquad ...(i)$

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $x + 2 = A(2x+2) + B \qquad ...(ii)$
$= 2Ax + 2A + B$

Comparing coefficients of x, $2A = 1 \Rightarrow A = \frac{1}{2}$

Putting A

Comparing constants, $2A + B = 2$

$= \frac{1}{2}$, $1 + B = 2 \Rightarrow B = 1$

Putting values of A and B in (*ii*), $x + 2 = \frac{1}{2}(2x + 2) + 1$

Putting this value of $(x + 2)$ in (*i*),

$$I = \int \frac{\frac{1}{2}(2x + 2) + 1}{\sqrt{x^2 + 2x + 3}}\,dx = \frac{1}{2}\int \frac{2x + 2}{\sqrt{x^2 + 2x + 3}}\,dx + \int \frac{dx}{\sqrt{x^2 + 2x + 3}}$$

$\Rightarrow \quad I = \frac{1}{2} I_1 + I_2 \quad ...(iii) \qquad I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$

Put $x^2 + 2x + 3 = t \quad \therefore \quad (2x+2) = \frac{dt}{dx} \Rightarrow (2x+2)\, dx = dt$

$I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \quad ...(iv)$

$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx = \int \frac{1}{\sqrt{x^2+2x+1+2}} dx$

$= \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}} dx = \log\left|x+1+\sqrt{(x+1)^2+(\sqrt{2})^2}\right|$

$\left[\because \int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x+\sqrt{x^2+a^2}\right|\right]$

$= \log\left|x+1+\sqrt{x^2+2x+3}\right| \quad ...(v)$

Putting values from (*iv*) and (*v*) in (*iii*),

$I = \sqrt{x^2+2x+3} + \log\left|x+1+\sqrt{x^2+2x+3}\right| + c.$

22. $\frac{x+3}{x^2-2x-5}$

Sol. Let $I = \int \frac{x+3}{x^2-2x-5} dx \quad ...(i)$

Let $x + 3 = A\frac{d}{dx}(x^2 - 2x - 5) + B$

or $\quad x + 3 = A(2x - 2) + B \quad ...(ii)$

$= 2Ax - 2A + B$

Comparing coefficients of x on both sides, $2A = 1 \Rightarrow A = \frac{1}{2}$

Comparing constants, $-2A + B = 3$

Putting $A = \frac{1}{2}$, $-1 + B = 3 \quad \Rightarrow \quad B = 4$

Putting values of A and B in (*ii*), $x + 3 = \frac{1}{2}(2x-2) + 4$

Putting this value in (*i*),

$\frac{1}{2}(2x-2)+4 \quad \frac{1}{}$

$$I = \int \frac{2_2}{x^2 - 2x - 5}\, dx = {}_2\int \frac{2x-2}{x^2-2x-5}\, dx + 4\int \frac{1}{x^2-2x-5}\, dx$$

$$= \frac{1}{2} I_1 + 4 I_2 \qquad \ldots(iii)$$

$$I_1 = \int \frac{2x-2}{x^2-2x-5}\, dx$$

Put $x^2 - 2x - 5 = t$. Therefore $(2x-2) = \frac{dt}{dx} \Rightarrow (2x-2)\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log|t| = \log|x^2 - 2x - 5|$ ...(*iv*)

Again $I_2 = \int \frac{1}{x^2 - 2x - 5} dx$

$= \int \frac{1}{x^2 - 2x + 1 - 1 - 5} dx = \int \frac{1}{(x-1)^2 - 6} dx$

$= \int \frac{1}{(x-1)^2 - 6} dx = \frac{1}{2\sqrt{6}} \log \left| \frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}} \right|$ ...(*v*)

$\left[\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right|\right]$

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*),

$I = \frac{1}{2} \log|x^2 - 2x - 5| + \frac{2}{\sqrt{6}} \log \left| \frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}} \right| + c.$

23. $\frac{5x + 3}{\sqrt{x^2 + 4x + 10}}$

Sol. Let $I = \int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}} dx$ ...(*i*)

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $5x + 3 = A(2x + 4) + B$ ...(*ii*)

$= 2Ax + 4A + B$

Comparing coefficients of x on both sides, $2A = 5 \Rightarrow A = \frac{5}{2}$

Comparing constants, $4A + B = 3$

Putting $A = \frac{5}{2}$, $10 + B = 3 \Rightarrow B = -7$

Putting values of A and B in (*ii*), $5x + 3 = \frac{5}{2}(2x + 4) - 7$

Putting this value in (*i*), $I = \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2 + 4x + 10}} dx$

$= \frac{5}{2} \int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}} dx - 7$

$$\int \frac{1}{\sqrt{x^2 + 4x + 10}}\, dx$$

or $\quad I = \frac{5}{2} I_1 - 7 I_2 \qquad \ldots(iii)$

$$I_1 = \int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}}\, dx$$

Put $x^2 + 4x + 10 = t$. Therefore $2x + 4 = \frac{dt}{dx} \Rightarrow (2x + 4)\, dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t}$$

$$= 2\sqrt{x^2 + 4x + 10} \qquad \text{...}(iv)$$

$$I_2 = \int \frac{1}{\sqrt{x^2 + 4x + 10}}\, dx = \int \frac{1}{\sqrt{x^2 + 4x + 4 + 6}}\, dx$$

$$= \int \frac{1}{\sqrt{(x+2)^2 + (\sqrt{6})^2}}\, dx = \log\left| x + 2 + \sqrt{(x+2)^2 + (\sqrt{6})^2} \right|$$

$$\left[\because \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log\left| x + \sqrt{x^2 + a^2} \right| \right]$$

$$= \log\left| x + 2 + \sqrt{x^2 + 4x + 10} \right| \qquad \text{...}(v)$$

Putting values of I_1 and I_2 from (iv) and (v) in (iii),

$$I = 5\sqrt{x^2 + 4x + 10} - 7\log\left| x + 2 + \sqrt{x^2 + 4x + 10} \right| + c.$$

Choose the correct answer in Exercises 24 and 25.

24. $\int \frac{dx}{x^2 + 2x + 2}$ **equals**

(A) $x\tan^{-1}(x+1) + C$ **(B)** $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1}x + C$ **(D)** $\tan^{-1}x + C.$

Sol. $\int \frac{dx}{x^2 + 2x + 2} = \int \frac{1}{x^2 + 2x + 1 + 1}\, dx = \int \frac{1}{(x+1)^2 + 1^2} dx$

$$= \frac{1}{1}\tan^{-1}\frac{(x+1)}{1} + C \qquad \left[\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \right]$$

$$= \tan^{-1}(x+1) + C$$

$\therefore$ Option (B) is the correct answer.

25. $\int \frac{dx}{\sqrt{9x - 4x^2}}$ **equals**

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$ **(B)** $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$ **(D)** $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{8}\right) + C.$

Sol. Let $I = \int \frac{dx}{\sqrt{9x - 4x^2}} = \int \frac{dx}{\sqrt{-4x^2 + 9x}}$ ...(*i*)

Here Quadratic expression is $-4x^2 + 9x = -4\left(x^2 - \frac{9}{4}x\right)$

$$= -4\left[x^2 - \frac{9}{4}x + \left(\frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right] = -4\left[\left(x - \frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]$$

$$= 4\left[-\left(x - \frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2\right] = 4\left[\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2\right]$$

Putting this value in (i),

$$I = \int \frac{1}{\sqrt{4\left[\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2\right]}}dx = \frac{1}{2}\int \frac{1}{\sqrt{\left[\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2\right]}}dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x - \frac{9}{8}}{\frac{9}{8}}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x - 9}{9}\right) + C \qquad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a}\right]$$

∴ Option (B) is the correct answer.

Exercise 7.5

Integrate the (rational) functions in Exercises 1 to 6:

1. $\dfrac{x}{(x+1)(x+2)}$

Sol. To integrate the (rational) function $\dfrac{x}{(x+1)(x+2)}$.

Let integrand $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. = $(x + 1)(x + 2)$,

$x = A(x + 2) + B(x + 1) \quad = Ax + 2A + Bx + B$

Comparing coefficients of x on both sides, $A + B = 1$ ...(*ii*)

Comparing constants, $2A + B = 0$ ...(*iii*)

Let us solve Eqns. (*ii*) and (*iii*) for A and B.

Eqn. (*iii*) – Eqn. (*ii*) gives, $A = -1$

Putting $A = -1$ in (*ii*), $-1 + B = 1 \Rightarrow B = 2$

Putting values of A and B in (*i*), $\dfrac{x}{(x+1)(x+2)} = \dfrac{-1}{x+1} + \dfrac{2}{x+2}$

$$\therefore \int \frac{x}{(x+1)(x+2)}\,dx = -\int \frac{1}{x+1}\,dx + 2\int \frac{1}{x+2}\,dx$$

$$= -\log|x+1| + 2\log|x+2| + c$$

$$= \log|x+2|^2 - \log|x+1| + c = \log\frac{(x+2)^2}{|x+1|} + c.$$

$(\because |t|^2 = t^2)$

2. $\dfrac{1}{x^2-9}$

Sol. To integrate the (rational) function $\dfrac{1}{x^2-9}$

$$\int \frac{1}{x^2-9}\,dx = \int \frac{1}{x^2-3^2}\,dx$$

$$= \frac{1}{2 \times 3} \log \left| \frac{x-3}{x+3} \right| + c \left[\because \int \frac{1}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \right]$$

$$= \frac{1}{6} \log \left| \frac{x-3}{x+3} \right| + c.$$

OR

Integrand $\frac{1}{x^2 - 9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

Now proceed as in the solution of Q.No.1.

3. $\dfrac{3x - 1}{(x - 1)(x - 2)(x - 3)}$.

Sol. To integrate the (rational) function $\dfrac{3x - 1}{(x - 1)(x - 2)(x - 3)}$

Let integrand $\dfrac{3x-1}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$ ...(*i*)

Multiplying by L.C.M. = $(x - 1)(x - 2)(x - 3)$, we have

$3x - 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$

$= A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$

$= Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C$

Comparing coefficients of x^2, x and constant terms on both sides, we have

Coefficients of x^2: $A + B + C = 0$ ...(*ii*)

Coefficient of x: $-5A - 4B - 3C = 3$ or $5A + 4B + 3C = -3$ ...(*iii*)

Constants: $6A + 3B + 2C = -1$ ...(*iv*)

Let us solve (*ii*), (*iii*) and (*iv*) for A, B, C.

Let us first form two Eqns. in two unknowns say A and B. Eqn. (*iii*) − 3 Eqn. (*i*) gives (to eliminate C),

$5A + 4B + 3C - 3A - 3B - 3C = -3$

or $2A + B = -3$ ...(*v*)

Eqn. (*iv*) − 2 Eqn. (*i*) gives (to eliminate C),

$6A + 3B + 2C - 2A - 2B - 2C = -1$

or $4A + B = -1$ ...(*vi*)

Eqn. (*vi*) − Eqn. (*v*) gives (to eliminate B),

$2A = -1 + 3 = 2 \Rightarrow A = \frac{2}{2} = 1.$

Putting A = 1 in (*v*), $2 + B = -3 \Rightarrow B = -5$

Putting A = 1 and B = − 5 in (*ii*), $1 - 5 + C = 0$

or $C - 4 = 0$ or $C = 4$

Putting values of A, B, C in (*i*),

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{4}{x-3}$$

$\therefore \int \frac{3x-1}{(x-1)(x-2)(x-3)}$

$= \int \frac{1}{x-1} dx - 5\int \frac{1}{x-2} dx + 4\int \frac{1}{x-3} dx$

$= \log |x - 1| - 5 \log |x - 2| + 4 \log |x - 3| + c.$

4. $\frac{x}{(x-1)(x-2)(x-3)}$

Sol. To integrate the (rational) function $\frac{x}{(x-1)(x-2)(x-3)}$.

Let integrand $\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$ ...(*i*)

(Partial fractions)

Multiplying by L.C.M. $= (x - 1)(x - 2)(x - 3)$,

$x = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$
$= A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$
$= Ax^2 - 5Ax + 6A + Bx^2 - 4Bx + 3B + Cx^2 - 3Cx + 2C$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2: $A + B + C = 0$ ...(*ii*)

x: $-5A - 4B - 3C = 1$ or $5A + 4B + 3C = -1$ ...(*iii*)

Constants: $6A + 3B + 2C = 0$ ...(*iv*)

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.

Let us first form two Eqns. in two unknowns say A and B.

Eqn. (*iii*) − 3 × Eqn. (*ii*) gives | To eliminate C

$5A + 4B + 3C - 3A - 3B - 3C = -1$ or $2A + B = -1$ ...(*v*)

Eqn. (*iv*) − 2 × Eqn. (*ii*) gives | To eliminate C

$4A + B = 0$ ...(*vi*)

Eqn. (*vi*) − Eqn. (*v*) gives (To eliminate B)

$2A = 1 \quad \therefore A = \frac{1}{2}$

Putting $A = \frac{1}{2}$ in (*v*), $1 + B = -1 \Rightarrow B = -2$

Putting $A = \frac{1}{2}$ and $B = -2$ in (*ii*),

$\frac{1}{2} - 2 + C = 0 \Rightarrow C = \frac{-1}{2} + 2 = \frac{-1+4}{2} = \frac{3}{2}$

Putting these values of A, B, C in (*i*), we have

$\frac{x}{(x-1)(x-2)(x-3)} = \frac{\frac{1}{2}}{x-1} - \frac{2}{x-2} + \frac{\frac{3}{2}}{x-3}$

$$\therefore \int \frac{\;}{(x-1)(x-2)(x-3)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\,dx - 2\int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x-3}\,dx$$

$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + c.$$

5. $\dfrac{2x}{x^2 + 3x + 2}$

Sol. To integrate the (rational) function $\dfrac{2x}{x^2 + 3x + 2}$.

Now $x^2 + 3x + 2 = x^2 + 2x + x + 2 = x(x + 2) + 1(x + 2)$
$= (x + 1)(x + 2)$

$\therefore$ Integrand $\dfrac{2x}{x^2 + 3x + 2} = \dfrac{2x}{(x+1)(x+2)}$

$$= \frac{A}{x+1} + \frac{B}{x+2} \quad ...(i)$$

(Partial Fractions)

Multiplying both sides by L.C.M. $= (x + 1)(x + 2)$,

$2x = A(x + 2) + B(x + 1) \quad = Ax + 2A + Bx + B$

Comparing coefficients of x and constant terms on both sides, we have

Coefficients of x: $A + B = 2$ ...(*ii*)

Constant terms: $2A + B = 0$ ...(*iii*)

Let us solve (*ii*) and (*iii*) for A and B.

(*iii*) – (*ii*) gives $A = -2$.

Putting $A = -2$ in (*ii*), $-2 + B = 2$. $\therefore$ $B = 4$

Putting values of A and B in (*i*), $\dfrac{2x}{x^2 + 3x + 2} = \dfrac{-2}{x+1} + \dfrac{4}{x+2}$

$$\therefore \int \frac{2x}{x^2 + 3x + 2}\, dx = -2\int \frac{1}{x+1}\, dx + 4\int \frac{1}{x+2}\, dx$$

$= -2 \log |x + 1| + 4 \log |x + 2| + c$

$= 4 \log |x + 2| - 2 \log |x + 1| + c$

Remark: Alternative method to evaluate $\displaystyle\int \frac{2x}{x^2 + 3x + 2} dx$

is $\displaystyle\int \frac{\text{Linear}}{\text{Quadratic}}\, dx$ as explained in solutions in Exercise 7.4

(Exercise 18 and Exercise 22.

6. $\dfrac{1 - x^2}{x(1 - 2x)}$

Sol. To integrate (rational) function $\dfrac{1 - x^2}{\ldots}$

$1 + x^2 \quad x(1 - 2x)$

$= \quad x - 2x^2 \quad = \quad \dfrac{-x^2+1}{-2x^2+x}$

[Here Degree of numerator = Degree of Denominator = 2
$\therefore$ We must divide numerator by denominator to make the degree of numerator smaller than degree of denominator so that we can form partial fractions.]

$$\begin{array}{r|l} -2x^2 + x \;)\; -x^2 + 1 \;(& \frac{1}{2} \\ -x^2 + \frac{x}{2} & \\ + \quad - & \\ \hline -\frac{x}{2} + 1 & \end{array}$$

$$\therefore \frac{1 - x^2}{x(1-2x)} = \text{Quotient} + \frac{\underline{\text{Remainder}}}{\text{Divisor}} = \frac{1}{2} + \frac{\left(-\frac{x}{2}+1\right)}{x(1-2x)}$$

$$\therefore \int \frac{1-x^2}{x(1-2x)}\,dx = \int \left(\frac{1}{2} + \frac{\left(-\frac{x}{2}+1\right)}{x(1-2x)} \right) dx$$

$$= \frac{1}{2}\int 1\,dx + \int \frac{-\frac{x}{2}+1}{x(1-2x)}\,dx \qquad \ldots(i)$$

Let integrand $\frac{-\frac{x}{2}+1}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$ ...(*ii*)

Multiplying by L.C.M. $= x(1-2x)$,

$$-\frac{x}{2} + 1 = A(1-2x) + Bx \qquad = A - 2Ax + Bx$$

Comparing coefficients of x, $-2A + B = \frac{-1}{2}$ ...(*iii*)

Comparing constants, $A = 1$ ...(*iv*)

Putting $A = 1$ from (*iv*) in (*iii*),

$$-2 + B = \frac{-1}{2} \Rightarrow B = \frac{-1}{2} + 2 = \frac{-1+4}{2} \quad \text{or} \quad B = \frac{3}{2}$$

Putting values of A and B in (*ii*),

$$\frac{-\frac{x}{2}+1}{x(1-2x)} = \frac{1}{x} + \frac{\frac{3}{2}}{1-2x}$$

$$\therefore \int \frac{-\frac{x}{2}+1}{x(1-2x)}\,dx = \int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{1}{1-2x}\,dx$$

$$= \log|x| + \frac{3}{2}$$

$$= \log|x| - \frac{3}{4}\log\frac{|1-2x|}{-2 \to \text{Coefficient of } x} + c$$

$$\log|1-2x| + c$$

Putting this value in (*i*),

$$\int \frac{1-x^2}{x(1-2x)}\,dx = \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + c.$$

Integrate the following functions in Exercises 7 to 12:

7. $\dfrac{x}{(x^2+1)(x-1)}$

Sol. To integrate the (rational) function $\dfrac{x}{(x^2+1)(x-1)}$.

Let integrand $\dfrac{x}{(x^2+1)(x-1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. = $(x^2+1)(x-1)$ on both sides,

$$x = (Ax+B)(x-1) + C(x^2+1)$$

$\Rightarrow$ $x = Ax^2 - Ax + Bx - B + Cx^2 + C,$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2 $\quad A + C = 0$ ...(*ii*)

x $\quad -A + B = 1$ ...(*iii*)

Constants $\quad -B + C = 0$ ...(*iv*)

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C

Adding (*ii*) and (*iii*) to eliminate A, B + C = 1 ...(*v*)

Adding (*iv*) and (*v*), $2C = 1 \Rightarrow C = \dfrac{1}{2}$

From (*iv*), $-B = -C \Rightarrow B = C = \dfrac{1}{2}$

From (*ii*), $A = -C = \dfrac{-1}{2}$

Putting these values of A, B, C in (*i*),

$$\frac{x}{(x^2+1)(x-1)} = \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x-1}$$

$$= \frac{-1}{2}\frac{x}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x^2+1} + \frac{1}{2}\frac{1}{x-1}$$

$$= \frac{-1}{4}\frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{x^2+1} + \frac{1}{2}\frac{1}{x-1}$$

$$\therefore \int \frac{x}{(x^2+1)(x-1)}\,dx$$

$$= \frac{-1}{4}\int\frac{2x}{x^2+1}\,dx + \frac{1}{2}\int\frac{1}{x^2+1}\,dx + \frac{1}{2}\int\frac{1}{x-1}\,dx$$

$$\Rightarrow \int\frac{x}{(x^2+1)(x-1)}\,dx = \frac{-1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x$$

$$+ \frac{1}{2}\log|x-1| + c \quad \left(\because \int\frac{f'(x)}{f(x)}dx = \log|f(x)|\right)$$

$$= \frac{-1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + c$$

$[\because x^2+1 > 0 \Rightarrow |x^2+1| = x^2+1]$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + c.$$

8. $\dfrac{x}{(x-1)^2(x+2)}$

Sol. To integrate the (rational) function $\dfrac{x}{(x-1)^2(x+2)}$.

Let integrand $\dfrac{x}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ ...(*i*)

(Partial fractions)

ultiplying both sides of (*i*) by L.C.M. = $(x-1)^2(x+2)$

$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

or $x = A(x^2 + 2x - x - 2) + B(x+2) + C(x^2 + 1 - 2x)$

or $x = Ax^2 + Ax - 2A + Bx + 2B + Cx^2 + C - 2Cx$

Comparing coefficients of x^2, x and constant terms on both sides

x^2 $A + C = 0$ ...(*ii*)

x $A + B - 2C = 1$ ...(*iii*)

Constants $-2A + 2B + C = 0$ ...(*iv*)

Let us solve (*ii*), (*iii*) and (*iv*) for A, B, C

From (*ii*), $A = -C$

Putting $A = -C$ in (*iv*), $2C + 2B + C = 0$

$\Rightarrow 2B = -3C \Rightarrow B = \dfrac{-3C}{2}$

Putting values of A and B in (*iii*),

$-C - \dfrac{-3C}{2} - 2C = 1 \Rightarrow -2C - 3C - 4C = 2$

$\Rightarrow -9C = 2 \Rightarrow C = \dfrac{-2}{9}$

Putting $C = \dfrac{-2}{9}$, $B = \dfrac{-3C}{2} = \dfrac{-3}{2}\left(\dfrac{-2}{9}\right) = \dfrac{1}{3}$ $\therefore A = -C = \dfrac{2}{9}$

Putting these values of A, B, C in (*i*),

$$\frac{x}{(x-1)^2(x+2)} = \frac{\frac{2}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} - \frac{\frac{2}{9}}{x+2}$$

$$\therefore \int \frac{x}{(x-1)^2(x+2)}\,dx$$

$$= \frac{2}{9}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int (x-1)^{-2}\,dx - \frac{2}{9}\int \frac{1}{x+2}\,dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\frac{(x-1)^{-1}}{(-1)(1)} - \frac{2}{9}\log|x+2| + c$$

$$= \frac{2}{9}(\log|x-1| - \log|x+2|) - \frac{1}{3(x} + c$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c.$$

9. $\dfrac{3x+5}{x^3 - x^2 - x + 1}$

Sol. To integrate the (rational) function $\dfrac{3x+5}{x^3 - x^2 - x + 1}$.

Now denominator $= x^3 - x^2 - x + 1$

$= x^2(x-1) - 1(x-1) = (x-1)(x^2-1)$

$= (x-1)(x-1)(x+1) = (x-1)^2(x+1)$

$\therefore$ Integrand $\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \quad ...(i) \text{ (Partial fractions)}$$

Multiplying by L.C.M. $= (x-1)^2(x+1)$,

$3x + 5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$= A(x^2-1) + B(x+1) + C(x^2+1-2x)$

$= Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$

Comparing coefficients of x^2, x and constant terms on both sides,

x^2 $\quad A + C = 0 \quad ...(ii)$

x $\quad B - 2C = 3 \quad ...(iii)$

Constants $\quad -A + B + C = 5 \quad ...(iv)$

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.

From (*ii*), $A = -C$ and from (*iii*), $B = 2C + 3$

Putting these values of A and B in (*iv*),

$$C + 2C + 3 + C = 5 \Rightarrow 4C = 2 \Rightarrow C = \frac{2}{4} = \frac{1}{2}$$

$\therefore \quad A = -C = -\frac{1}{2}$

and $\quad B = 2C + 3 = 2\left(\frac{1}{2}\right) + 3 = 1 + 3 = 4.$

Putting these values of A, B, C in (*i*)

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{\frac{-1}{2}}{x-1} + \frac{4}{(x-1)^2} + \frac{\frac{1}{2}}{x+1}$$

$$\therefore \int \frac{3x+5}{x^3-x^2-x+1}\,dx$$

$$= \frac{-1}{2}\int \frac{1}{x-1}\,dx + 4\int (x-1)^{-2}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{-1}{2}\log|x-1| + 4\frac{(x-1)^{-1}}{(-1)(1)} + \frac{1}{2}\log|x+1| + c$$

(1) ↓ Coeff. of x

$$= \frac{1}{2}(\log|x+1| - \log|x-1|) - \frac{4}{x-1} + c$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + c.$$

10. $\dfrac{2x-3}{(x^2-1)(2x+3)}$

Sol. To integrate the rational function $\dfrac{2x-3}{(x^2-1)(2x+3)}$.

Let integrand $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)}$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3} \quad \ldots(i)$$

Multiplying both sides by L.C.M. = $(x - 1)(x + 1)(2x + 3)$,

$2x - 3 = A(x + 1)(2x + 3) + B(x - 1)(2x + 3) + C(x - 1)(x + 1)$

or $2x - 3 = A(2x^2 + \mathbf{3x + 2x} + 3) + B(2x^2 + \mathbf{3x - 2x} - 3) + C(x^2 - 1)$

Comparing coefficients of x^2, x and constant terms on both sides,

x^2 $\quad 2A + 2B + C = 0 \quad \ldots(ii)$

x $\quad 5A + B = 2 \quad \ldots(iii)$

Constants $\quad 3A - 3B - C = -3 \quad \ldots(iv)$

Let us solve Eqns. (*ii*), (*iii*) and (*iv*) for A, B, C.

Eqn. (*ii*) + Eqn. (*iv*) gives (to eliminate C)

$$5A - B = -3 \quad \ldots(v)$$

Adding Eqns. (*iii*) and (*v*), $10A = -1 \Rightarrow A = \frac{-1}{10}$

Putting $A = \frac{-1}{10}$ in (*iii*), $\frac{-5}{10} + B = 2 \Rightarrow B = 2 + \frac{1}{2} = \frac{5}{2}$

Putting values of A and B in (*ii*),

$\frac{-1}{5} + 5 + C = 0 \quad \therefore C = \frac{1}{5} - 5 = \frac{1-25}{25} = \frac{-24}{5}$

Putting values of A, B, C in (*i*),

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{\frac{-1}{10}}{x-1} + \frac{\frac{5}{2}}{x+1} - \frac{\frac{24}{5}}{2x+3}$$

$$\therefore \int \frac{2x-3}{(x^2-1)(2x+3)}\,dx$$

$$= \frac{-1}{10}\int \frac{1}{x-1}\,dx + \frac{5}{2}\int \frac{1}{x+1}\,dx - \frac{24}{5}\int \frac{1}{2x+3}\,dx$$

$$= \frac{-1}{10}\,\frac{\log|x-1|}{1 \to \text{Coeff. of } x} + \frac{5}{2}\,\frac{\log|x+1|}{1} - \frac{24}{5}\,\frac{\log|2x+3|}{2 \to \text{Coeff. of } x} + c$$

$$= \frac{-1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + c$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + c.$$

11. $\dfrac{5x}{(x+1)(x^2-4)}$

Sol. To integrate the rational function $\dfrac{5x}{(x+1)(x^2-4)}$.

Let integrand $\dfrac{5x}{(x+1)(x^2-4)} = \dfrac{5x}{(x+1)(x+2)(x-2)}$

$$= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-2} \quad \ldots(i) \quad \text{(Partial fractions)}$$

Multiplying both sides of (*i*) by L.C.M.

$= (x + 1)(x + 2)(x - 2),$

$5x = A(x + 2)(x - 2) + B(x + 1)(x - 2) + C(x + 1)(x + 2)$

$= A(x^2 - 4) + B(x^2 - x - 2) + C(x^2 + 3x + 2)$

$= Ax^2 - 4A + Bx^2 - Bx - 2B + Cx^2 + 3Cx + 2C.$

Comparing coefficients of x^2, x and constant terms on both sides,

x^2 $A + B + C = 0$ ...(*ii*)

x $- B + 3C = 5$ ...(*iii*)

Constants $- 4A - 2B + 2C = 0$

Dividing by $- 2$, $2A + B - C = 0$ ...(*iv*)

Let us solve (*ii*), (*iii*) and (*iv*) for A, B, C

Eqn. (*ii*) × 2 – Eqn. (*iv*) gives (To eliminate A) because Eqn. (*iii*) does not involve A.

$2A + 2B + 2C - (2A + B - C) = 0,$

i.e., $2A + 2B + 2C - 2A - B + C = 0$

$\Rightarrow$ $B + 3C = 0$ ...(*v*)

Adding Eqns. (*iii*) and (*v*),

$$6C = 5 \quad \Rightarrow \quad C = \frac{5}{6}$$

Putting $C = \frac{5}{6}$ in (*iii*), $- B + \frac{15}{6} = 5 \quad \Rightarrow \quad - B = 5 - \frac{15}{6}$

$$\Rightarrow \quad - B = \frac{30 - 15}{6} = \frac{15}{6} = \frac{5}{2} \quad \Rightarrow \quad B = \frac{-5}{2}$$

Putting $B = \frac{-5}{2}$ and $C = \frac{5}{6}$ in (*ii*), $A - \frac{5}{2} + \frac{5}{6} = 0$

$$\Rightarrow \quad A = \frac{5}{2} - \frac{5}{6} = \frac{15 - 5}{6} = \frac{10}{6} = \frac{5}{3}$$

Putting values of A, B, C in (*i*),

$$\frac{5x}{(x + 1)(x^2 - 4)} = \frac{\frac{5}{3}}{x + 1} - \frac{\frac{5}{2}}{x + 2} + \frac{\frac{5}{6}}{x - 2}$$

$$\therefore \int \frac{5x}{(x + 1)(x^2 - 4)} dx = \frac{5}{3}\int \frac{1}{x + 1} dx - \frac{5}{2}\int \frac{1}{x + 2} dx + \frac{5}{6}\int \frac{1}{x - 2} dx$$

$$= \frac{5}{3} \log |x + 1| - \frac{5}{2} \log |x + 2| + \frac{5}{6} \log |x - 2| + c.$$

12. $\frac{x^3 + x + 1}{x^2 - 1}$

Sol. Here degree of numerator is greater than degree of denominator.

Therefore, dividing the numerator by the denominator,

$$\begin{array}{r|l} x^2-1 & x^3+x+1 \;(\; x \\ & \underline{x^3 - x} \\ & \;\; 2x+1 \end{array}$$

$$\therefore \quad \frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1} \qquad \ldots(i)$$

$$\left[\textbf{Rational function = Quotient + }\frac{\textbf{Remainder}}{\textbf{Divisor}}\right]$$

Let $\frac{2x+1}{x^2-1} = \frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$ ...(*ii*)

Multiplying by L.C.M. = $(x+1)(x-1)$, we have

$2x + 1 = A(x - 1) + B(x + 1)$

or $2x + 1 = Ax - A + Bx + B$

By equating the coefficients of x and constant terms, we get

$A + B = 2$ ...(*iii*)

and $-A + B = 1$ ...(*iv*)

(*iii*) + (*iv*) gives $2B = 3 \Rightarrow B = \frac{3}{2}$

Putting $B = \frac{3}{2}$ in (*iii*), we get $A + \frac{3}{2} = 2$ or $A = \frac{1}{2}$

Putting values of A and B in eqn. (*ii*), we have

$$\frac{2x+1}{x^2-1} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}}{x-1}$$

Putting this value of $\frac{2x+1}{x^2-1}$ in (*i*),

$$\frac{x^3+x+1}{x^2-1} = x + \frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}}{x-1}$$

$$\therefore \int \frac{x^3+x+1}{x^2-1}\,dx = \int x\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + c.$$

Integrate the following functions in Exercises 13 to 17:

13. $\frac{2}{(1-x)(1+x^2)}$

Sol. To find integral of the Rational function $\frac{2}{(1-x)(1+x^2)}$.

Let integrand $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$ ...(*i*)

(Partial Fractions)

Multiplying by L.C.M. $= (1 - x)(1 + x^2)$

$$2 = A(1 + x^2) + (Bx + C)(1 - x)$$

or $$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Comparing coefficients of x^2, x and constant terms, we have

$\boldsymbol{x^2}$ $A - B = 0$ ...(*ii*)

$\boldsymbol{x}$ $B - C = 0$ ...(*iii*)

Constant terms $A + C = 2$ ...(*iv*)

Let us solve (*ii*), (*iii*), (*iv*) for A, B, C

From (*ii*), A = B and from (*iii*), B = C

$\therefore$ A = B = C

Putting A = C in (*iv*), C + C = 2 or 2C = 2 or C = 1

$\therefore$ A = C = 1 $\therefore$ B = A = 1

Putting these values of A, B, C in (*i*),

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2} = \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}$$

$$= \frac{1}{1-x} + \frac{1}{2}\frac{2x}{1+x^2} + \frac{1}{1+x^2}$$

$$\therefore \int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= \frac{\log|1-x|}{-1 \to \text{Coefficient of } x} + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + c$$

$$\left[\because \int \frac{2x}{1+x^2}dx = \int \frac{f'(x)}{f(x)}dx = \log|f(x)|\right]$$

$$= -\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + c$$

$(\because 1 + x^2 > 0$, therefore $|1 + x^2| = 1 + x^2)$

Note. $\log|1-x| = \log|-(x-1)|$

$= \log|x-1|$ because $|-t| = |t|$.

14. $\dfrac{3x-1}{(x+2)^2}$

Sol. To find integral of rational function $\dfrac{3x-1}{(x+2)^2}$.

Let $I = \displaystyle\int \frac{3x-1}{(x+2)^2}dx$ ...(*i*)

Form $\displaystyle\int \frac{\text{Polynomial function}}{(\text{Linear})^k}dx$ where k is a positive integer,

put Linear = t.

Here put $x + 2 = t$ $\Rightarrow x = t - 2$

$\therefore \dfrac{dx}{dt} = 1 \Rightarrow dx = dt$

Putting these values in (*i*),

$$I = \int \frac{3(t-2)-1}{t^2}\,dt = \int \frac{3t-6-1}{t^2}\,dt = \int \frac{3t-7}{t^2}\,dt$$

$$= \int \left(\frac{3t}{t^2} - \frac{7}{t^2}\right) dt = \int \left(\frac{3}{t} - \frac{7}{t^2}\right) dt$$

$$= 3\int \frac{1}{t}\,dt - 7\int t^{-2}\,dt = 3\log|t| - 7\frac{t^{-1}}{-1} + c$$

$$= 3\log|t| + \frac{7}{t} + c$$

Putting $t = x + 2$, $= 3 \log |x + 2| + \frac{7}{x+2} + c.$

Remark. Alternative solution is Let $\frac{3x-1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$.

15. $\frac{1}{x^4 - 1}$

Sol. To find integral of $\frac{1}{x^4-1}$.

Let integrand $\frac{1}{x^4 - 1} = \frac{1}{(x^2-1)(x^2+1)}$.

Put $x^2 = y$ **only** to form partial fractions.

$$= \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} \quad ...(i)$$

Multiplying by L.C.M. $= (y-1)(y+1)$

$1 = A(y + 1) + B(y - 1)$ or $1 = Ay + A + By - B$

Comparing coeffs. of y and constant terms, we have

Coefficients of y: $A + B = 0$ ...(*ii*)

Constant terms $A - B = 1$ ...(*iii*)

Adding (*ii*) and (*iii*), $2A = 1$ $\Rightarrow A = \frac{1}{2}$

Putting $A = \frac{1}{2}$ in (*ii*), $\frac{1}{2} + B = 0$ $\Rightarrow B = \frac{-1}{2}$

Putting values of A, B **and** y in (*i*),

$$\frac{1}{x^4-1} = \frac{\frac{1}{2}}{x^2-1} - \frac{\frac{1}{2}}{x^2+1}$$

$$\therefore \int \frac{1}{x^4-1} dx = \frac{1}{2}\int \frac{1}{x^2-1^2} dx - \frac{1}{2}\int \frac{1}{x^2+1} dx$$

$$= \frac{1}{2}\cdot\frac{1}{2.1} \log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1} x + c$$

$$\left[\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

Note. Must put $y = x^2$ in (i) along with values of A and B before writing values of integrals.

Remark. Alternative solution is:

$$\frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

But the above given solution is better.

16. $\dfrac{1}{x(x^n+1)}$

Sol. Let $I = \displaystyle\int \frac{1}{x(x^n+1)}\,dx$

Multiplying both numerator and denominator of integrand by nx^{n-1}. $\left[\because \dfrac{d}{dx}(x^n+1) = nx^{n-1}\right]$

$$I = \int \frac{nx^{n-1}}{n\,x^{n-1}\,x(x^n+1)}\,dx = \frac{1}{n}\int \frac{n\,x^{n-1}}{x^n\,(x^n+1)}\,dx \quad \ldots(i)$$

$(\because\ n-1+1=n)$

Put $x^n = t$. Therefore $n\,x^{n-1} = \dfrac{dt}{dx}$. $\therefore\ n\,x^{n-1}\,dx = dt.$

$\therefore$ From (i), $I = \dfrac{1}{n}\displaystyle\int \frac{dt}{t(t+1)} = \frac{1}{n}\int \frac{1}{t(t+1)}dt$

Adding and subtracting t in the numerator of integrand,

$$= \frac{1}{n}\int \frac{t+1-t}{t(t+1)}\,dt = \frac{1}{n}\int\left(\frac{t+1}{t(t+1)} - \frac{t}{t(t+1)}\right)dt \quad \left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \frac{1}{n}\left[\int \frac{1}{t}\,dt - \int\frac{1}{t+1}\,dt\right] = \frac{1}{n}[\log|t| - \log|t+1| + c]$$

$$= \frac{1}{n}\log\left|\frac{t}{t+1}\right| + c$$

Putting $t = x^n$, $= \dfrac{1}{n}\log\left|\dfrac{x^n}{x^n+1}\right| + c$

Remark: Alternative solution for $\displaystyle\int \frac{1}{t(t+1)}\,dt$ is:

Let $\dfrac{1}{t(t+1)} = \dfrac{A}{t} + \dfrac{B}{t+1}$.

But the above given solution is better.

17. $\dfrac{\cos x}{(1-\sin x)(2-\sin x)}$

Sol. Let $I = \displaystyle\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx \quad \ldots(i)$

Put $\sin x = t$. Therefore $\cos x = \dfrac{dt}{dx}$ $\Rightarrow$ $\cos x\,dx = dt,$

$\therefore$ From (i), $\int \frac{1}{(1-t)(2-t)}\, dt = \int \frac{(2-t)-(1-t)}{(1-t)(2-t)}\, dt$

[$\because$ Difference of two factors in the denominator namely $1-t$ and $2-t$ is $(2-t)-(1-t) = 2-t-1+t = 1$]

$$= \int \left(\frac{2-t}{(1-t)(2-t)} - \frac{(1-t)}{(1-t)(2-t)} \right) dt \quad \left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c} \right]$$

$$= \int \left(\frac{1}{1-t} - \frac{1}{2-t}\right) dt = \int \frac{1}{1-t}\, dt - \int \frac{1}{2-t}\, dt$$

$$= \frac{\log|1-t|}{-1 \rightarrow \text{Coefficient of } t} - \frac{\log|2-t|}{-1} + c$$

$$= -\log|1-t| + \log|2-t| + c$$

$$= \log|2-t| - \log|1-t| + c \quad = \log\left|\frac{2-t}{1-t}\right| + c$$

Putting $t = \sin x$, $= \log\left|\frac{2-\sin x}{1-\sin x}\right| + c$

Remark: Alternative solution for $\int \frac{1}{(1-t)(2-t)}\, dt$ is

Let $\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t}$

Integrate the following functions for Exercises 18 to 21:

18. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Sol. To integrate the rational function $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ .... (i)

Put $x^2 = y$ in the integrand to get

$$= \frac{(y+1)(y+2)}{(y+3)(y+4)} = \frac{y^2+3y+2}{y^2+7y+12} \qquad ...(ii)$$

Here degree of numerator = degree of denominator (= 2)
So have to perform long division to make the degree of numerator smaller than degree of denominator so that the concept of forming partial fractions becomes valid.

$$y^2 + 7y + 12 \overline{)\, y^2 + 3y + 2\,(} \; 1$$
$$\underline{\begin{array}{ccc} y^2 & +\,7y & +\,12 \\ - & - & - \end{array}}$$
$$-\,4y - 10$$

$\therefore$ From (i) and (ii),

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \frac{(-4y-10)}{(y+3)(y+4)} \qquad ...(iii)$$

Let us form partial fractions of $\frac{(-4y-10)}{(y+3)(y+4)}$.

Let $\frac{-4y-10}{(y+3)(y+4)} = \frac{A}{y+3} + \frac{B}{y+4}$ ...(*iv*)

Multiplying by L.C.M. = $(y + 3)(y + 4)$

$-4y - 10 = A(y + 4) + B(y + 3) = Ay + 4A + By + 3B$

Comparing coefficients of y, $A + B = -4$ ...(*v*)

Comparing constants, $4A + 3B = -10$ ...(*vi*)

Let us solve Eqns. (*v*) and (*vi*) for A and B.

Eqn. (*v*) × 4 gives, $4A + 4B = -16$ ...(*vii*)

Eqn. (*vi*) – Eqn. (*vii*) gives, – B = 6 or B = – 6.
Putting B = – 6 in (*v*), A – 6 = – 4 ⇒ A = – 4 + 6 = 2
Putting these values of A and B in (*iv*),

$$\frac{-4y-10}{(y+3)(y+4)} = \frac{2}{y+3} - \frac{6}{y+4}$$

Putting this value in (*iii*),

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{2}{y+3} - \frac{6}{y+4}$$

In R.H.S., Putting $y = x^2$ (before integration)

$$= 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}$$

$$\therefore \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$$

$$= \int 1\,dx + 2\int \frac{1}{x^2+(\sqrt{3})^2}\,dx - 6\int \frac{1}{x^2+2^2}\,dx$$

$$= x + 2\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 6\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + c.$$

19. $\dfrac{2x}{(x^2+1)(x^2+3)}$

Sol. Let $I = \int \frac{2x}{(x^2+1)(x^2+3)}dx$

Put $x^2 = t$. Differentiating both sides $2x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{(t+1)(t+3)}$$

Dividing and multiplying by 2,

$(\because (t+3)-(t+1) = t+3-t-1 = 2)$

$$= \frac{1}{2}\int \frac{2}{(t+1)(t+3)}dt = \frac{1}{2}\int \frac{(t+3)-(t+1)}{(t+1)(t+3)}dt$$

$$= \frac{1}{2}\int\left(\frac{1}{t+1} - \frac{1}{t+3}\right)dt = \frac{1}{2}[\log|t+1| - \log|t+3|] + c$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + c = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + c = \frac{1}{2}\log\left(\frac{x^2+1}{x^2+3}\right) + c.$$

20. $\dfrac{1}{x(x^4-1)}$

Sol. Let I = $\displaystyle\int \frac{1}{x(x^4-1)}\,dx$

Multiplying both numerator and denominator of integrand by $4x^3$. $\left(\because \dfrac{d}{dx}(x^4-1) = 4x^3\right)$

$$I = \int \frac{4x^3}{4x^4(x^4-1)}dx = \frac{1}{4}\int \frac{4x^3}{x^4(x^4-1)}dx \qquad ...(i)$$

Put $x^4 = t$. Therefore $4x^3 = \frac{dt}{dx} \Rightarrow 4x^3\,dx = dt$.

$\therefore$ From (i), $I = \frac{1}{4}\int \frac{dt}{t(t-1)} = \frac{1}{4}\int \frac{t-(t-1)}{t(t-1)}dt$

$[\because t-(t-1) = t-t+1 = 1]$

$$= \frac{1}{4}\int\left(\frac{t}{t(t-1)} - \frac{(t-1)}{t(t-1)}\right)dt = \frac{1}{4}\int\left(\frac{1}{t-1} - \frac{1}{t}\right)dt$$

$$= \frac{1}{4}\left[\int \frac{1}{t-1}dt - \int \frac{1}{t}dt\right] = \frac{1}{4}[\log|t-1| - \log|t|] + c$$

$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + c$$

Putting $t = x^4$, $= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + c.$

Remark: Alternative solution is:

$$\frac{1}{x(x^4-1)} = \frac{1}{x(x^2-1)(x^2+1)} = \frac{1}{x(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{Dx+E}{x^2+1}$$

But the solution given above is much better.

21. $\frac{1}{(e^x-1)}$

Sol. Let $I = \int \frac{1}{e^x-1}dx \qquad ...(i)$

Put $e^x = t$. Therefore $e^x = \frac{dt}{dx} \Rightarrow e^x\,dx = dt \Rightarrow dx = \frac{dt}{e^x}$

$\left(\textbf{Rule to evaluate} \int f(e^x)\,dx\textbf{, put } e^x = t\right)$

$\therefore$ From (i), $I = \int \frac{1}{t-1}\,\frac{dt}{}$

$e^x =$ $\frac{1}{dt}$ $t-1$ $\frac{1}{t(t-1)}$ $\int$ $\int$

$t =$ dt

$$= \int \frac{t-(t-1)}{t(t-1)}\, dt = \int \left(\frac{t}{t(t-1)} - \frac{(t-1)}{t(t-1)}\right) dt = \int \frac{1}{t-1}\, dt - \int \frac{1}{t}\, dt$$

$$= \log |t-1| - \log |t| + c = \log \left|\frac{t-1}{t}\right| + c.$$

Putting $t = e^x$, $= \log \left|\frac{e^x - 1}{e^x}\right| + c$.

Choose the correct answer in each of the Exercises 22 and 23:

22. $\int \frac{x\,dx}{(x-1)(x-2)}$ **equals**

(A) $\log \left| \frac{(x-1)^2}{x-2} \right| + C$ (B) $\log \left| \frac{(x-2)^2}{x-1} \right| + C$

(C) $\log \left| \left(\frac{x-1}{x-2}\right)^2 \right| + C$ (D) $\log |(x-1)(x-2)| + C.$

Sol. Let integrand $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$ ...(*i*)

(Partial fractions)

Multiplying by L.C.M. = $(x-1)(x-2)$,

$x = A(x-2) + B(x-1)$

$= Ax - 2A + Bx - B$

Comparing coefficients of x and constant terms on both sides,

Coefficients of x: $A + B = 1$ ...(*ii*)

Constant terms: $-2A - B = 0$ ...(*iii*)

Let us solve (*ii*) and (*iii*) for A and B

Adding (*ii*) and (*iii*), $-A = 1$ or $A = -1$

Putting $A = -1$ in (*ii*) $-1 + B = 1$ or $B = 2$

Putting values of A and B in (*i*),

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}$$

$$\therefore \int \frac{x}{(x-1)(x-2)}\,dx = -\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-2}\,dx$$

$= -\log|x-1| + 2\log|x-2| + c$

$= \log|(x-2)^2| - \log|x-1| + c$ $(\because n \log m = \log m^n)$

$$= \log \left| \frac{(x-2)^2}{x-1} \right| + c$$

$\therefore$ Option (B) is the correct answer.

23. $\int \frac{dx}{x(x^2+1)}$ **equals**

(A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

(B) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(C) $-\log|x| + {}_{2}\log(x^2+1) + C$

(D) $\frac{1}{2}\log|x| + \log(x^2+1) + C.$

Sol. Let I = $\int \frac{1}{x(x^2+1)}dx$

Multiplying both numerator and denominator of integrand by $2x$, $\left(\because \frac{d}{dx}(x^2+1) = 2x\right)$

$$\Rightarrow \quad I = \int \frac{2x}{2x^2(x^2+1)}\,dx \qquad \ldots(i)$$

Put $x^2 = t$. $\quad \therefore \quad 2x = \frac{dt}{dx} \quad \Rightarrow \quad 2x\,dx = dt$

$\therefore$ From (i), $I = \int \frac{dt}{2t(t+1)} = \frac{1}{2}\int \frac{1}{t(t+1)}\,dt$

Adding and subtracting t in the numerator of integrand,

$$= \frac{1}{2}\int \frac{(t+1)-t}{t(t+1)}\,dt = \frac{1}{2}\int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$= \frac{1}{2}(\log|t| - \log|t+1|) + c$$

Putting $t = x^2$, $I = \frac{1}{2}(\log|x^2| - \log|x^2+1|) + c$

$$= \frac{1}{2}(2\log|x| - \log(x^2+1)) + c$$

$(\because\ x^2 + 1 \geq 1 > 0$ and hence $|x^2+1| = x^2+1)$

$$= \log|x| - \frac{1}{2}\log(x^2+1) + c$$

$\therefore$ Option (A) is the correct answer.

Exercise 7.6

Integrate the functions in Exercises 1 to 8:

1. x sin x

Sol. $\int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\, dx$

Applying Product Rule $\text{I} \int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx \right) dx$

$= x \int \sin x\, dx - \int \left(\frac{d}{dx}(x) \int \sin x\, dx \right) dx$

$= x(-\cos x) - \int 1(-\cos x)\, dx = -x\cos x - \int -\cos x\, dx$

$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + c$

Note. $\int \sin x\, dx = -\cos x$.

2. x sin 3x

Sol. $\int \underset{\text{I}}{x} \underset{\text{II}}{\sin 3x}\, dx$

Applying Product Rule $\text{I} \int \text{II}\, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II}\, dx \right) dx$

$= x \int \sin 3x\, dx - \int \left(\frac{d}{dx}(x) \int \sin 3x\, dx \right) dx$

$= x\left(\frac{-\cos 3x}{3} \right) - \int \left[1\left(\frac{-\cos 3x}{3} \right) \right] dx + c$

$= \frac{-1}{3} x\cos 3x + \frac{1}{3} \int \cos 3x\, dx + c$

$= \frac{-1}{3} x \cos 3x + \frac{1}{3} \frac{\sin 3x}{3} + c = \frac{-1}{3} x \cos 3x + \frac{1}{9} \sin 3x + c.$

3. $x^2 e^x$

Sol. $\int \underset{I}{x^2} \underset{II}{e^x} \, dx$

Applying Product Rule I $\int \text{II}\, dx - \int \left[\left(\frac{d}{dx}(\text{I})\right) \int \text{II}\, dx\right] dx$

$= x^2 \int e^x dx - \int \left[\left(\frac{d}{dx} x^2\right) \int e^x dx\right] dx = x^2 e^x - \int 2x\, e^x\, dx$

$= x^2 e^x - 2 \int \underset{I}{x} \underset{II}{e^x} \, dx$

Again Applying Product Rule

$= x^2 e^x - 2 \left[x \int e^x dx - \int \left[\frac{d}{dx}(x) \int e^x dx\right] dx\right]$

$= x^2 e^x - 2 \left(x e^x - \int 1.e^x dx\right) = x^2 e^x - 2 \left(x e^x - \int e^x dx\right)$

$= x^2 e^x - 2x e^x + 2 \int e^x dx + c = x^2 e^x - 2x e^x + 2e^x + c$

$= e^x (x^2 - 2x + 2) + c.$

4. $x \log x$

Sol. $\int x \log x \, dx = \int \underset{I}{(\log x)} . \underset{II}{x} \, dx$

Applying Product Rule I $\int \text{II}\, dx - \int \left[\frac{d}{dx}(\text{I}) \int \text{II}\, dx\right] dx$

$= (\log x) \int x \, dx - \int \left[\frac{d}{dx}(\log x) \int x\, dx\right] dx$

$= (\log x) \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2} dx = \frac{1}{2} x^2 \log x - \frac{1}{2} \int x \, dx$

$\left(\because \frac{x^2}{x} = \frac{x.x}{x} = x\right)$

$= \frac{1}{2} x^2 \log x - \frac{1}{2} \frac{x^2}{2} + c = \frac{x^2}{2} \log x - \frac{x^2}{4} + c.$

5. x x

Sol. $\int x \log 2x \, dx = \int \underset{\text{I}}{(\log 2x)} \, . \underset{\text{II}}{x} \, dx$

Applying Product Rule $\text{I} \int \text{II} \, dx - \int \left(\frac{d}{dx}(\text{I}) \int \text{II} \, dx \right) dx$

$= (\log 2x) \int x \, dx - \int \left(\frac{d}{dx} (\log 2x) \int x \, dx \right) dx$

$= (\log 2x) \frac{x^2}{2} - \int \frac{1}{2x} \, . \, 2 \, . \, \frac{x^2}{2} \, dx$

$= \frac{1}{2} x^2 \log 2x - \frac{1}{2} \int x \, dx$ $\left[\because \frac{x^2}{x} = \frac{x \cdot x}{x} = x\right]$

$= \frac{1}{2} x^2 \log 2x - \frac{1}{2} \frac{x^2}{2} + c = \frac{x^2}{2} \log 2x - \frac{x^2}{4} + c.$

6. $x^2 \log x$

Sol. $\int x^2 \log x \, dx = \int \underset{\text{I}}{(\log x)} \underset{\text{II}}{x^2} \, dx$

Applying Product Rule: $\text{I} \int \text{II} \, dx - \int \left(\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right) dx$

$= \log x \int x^2 \, dx - \int \left(\frac{d}{dx} (\log x) \int x^2 \, dx \right) dx$

$= (\log x) \frac{x^3}{3} - \int \frac{1}{x} \frac{x^3}{3} dx = \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 \, dx$ $\left[\because \frac{x^3}{x} = x^2\right]$

$= \frac{x^3}{3} \log x - \frac{1}{3} \frac{x^3}{3} + c = \frac{x^3}{3} \log x - \frac{x^3}{9} + c.$

7. $x \sin^{-1} x$

Sol. Let $\text{I} = \int x \sin^{-1} x \, dx.$

Put $x = \sin \theta$. Differentiating both sides $dx = \cos \theta \, d\theta$

$\therefore \quad \text{I} = \int \sin \theta \cdot \theta \cdot \cos \theta \, d\theta = \frac{1}{2} \int \theta \cdot 2 \sin \theta \cos \theta \, d\theta$

$= \frac{1}{2} \int \underset{\text{I}}{\theta} \; \underset{\text{II}}{\sin 2\theta} \, d\theta$

Integrating by parts

$= \frac{1}{2} \left[\theta \left(-\frac{\cos 2\theta}{2} \right) - \int 1 \cdot \left(-\frac{\cos 2\theta}{2} \right) d\theta \right]$

$= \frac{1}{4} \left[-\theta \cos 2\theta + \int \cos 2\theta \, d\theta \right] = \frac{1}{4} \left[-\theta \cos 2\theta + \frac{\sin 2\theta}{2} \right] + c$

$= \frac{1}{4} [-\theta (1 - 2 \sin^2 \theta) + \sin \theta \cos \theta] + c$

$(\because \sin 2\theta = 2 \sin \theta \cos \theta)$

$= \frac{1}{4} [-\sin^{-1} x \cdot (1 - 2x^2) + x \sqrt{1 - x^2}] + c$

$\left[\because \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2}\right]$

$$= \frac{1}{4}(2x^2 - 1)\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + c.$$

8. $x \tan^{-1} x$

Sol. Let $I = \int x \tan^{-1} x \, dx = \int (\tan^{-1} x) \, . \, x \, dx$

$$= (\tan^{-1} x) \, . \, \frac{x^2}{2} - \int \frac{1}{1+x^2} \, . \, \frac{x^2}{2} dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx \qquad \left[\because \frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = 1 - \frac{1}{1+x^2}\right]$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + c$$

$$= \frac{1}{2}[x^2 \tan^{-1} x - x + \tan^{-1} x] + c = \frac{1}{2}[(x^2+1)\tan^{-1} x - x] + c.$$

9. Integrate the functions in Exercises 9 to 15:
$x \cos^{-1} x$

Sol. Let $I = \int x \cos^{-1} x \; dx$ ...(*i*)

Put $\cos^{-1} x = \theta$. Therefore $x = \cos\theta$.

$\therefore \frac{dx}{d\theta} = -\sin\theta \Rightarrow dx = -\sin\theta\, d\theta$

$\therefore$ From(*i*), $I = \int (\cos\theta)\,\theta\,(-\sin\theta\, d\theta) = \frac{-1}{2}\int \theta\,(2\sin\theta\cos\theta)\,d\theta$

$$= \frac{-1}{2}\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sin 2\theta}\, d\theta$$

Applying Product Rule: $\int \text{I}\ \text{II}\ d\theta - \int \left[\frac{d}{d\theta}(\text{I})\int \text{II}\ d\theta\right] d\theta$

$$= \frac{-1}{2}\left[\theta\left(\frac{-\cos 2\theta}{2}\right) - \int 1\left(\frac{-\cos 2\theta}{2}\right) d\theta\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{2}\theta\cos 2\theta + \frac{1}{2}\int \cos 2\theta\, d\theta\right] = \frac{1}{4}\theta\cos 2\theta - \frac{1}{4}\left(\frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{1}{4}\theta\cos 2\theta - \frac{1}{8}(2\sin\theta\cos\theta) + c$$

$$= \frac{1}{4}\theta\,(2\cos^2\theta - 1) - \frac{1}{4}\sqrt{1-\cos^2\theta}\ .\cos\theta + c$$

Putting $\cos\theta = x$ and $\theta = \cos^{-1} x$;

$$= \frac{1}{4}(\cos^{-1} x)(2x^2 - 1) - \frac{1}{4}\sqrt{1-x^2}\ .\ x + c$$

$$= (2x^2 - 1)\frac{\cos^{-1} x}{4} - \frac{x}{4}\sqrt{1 - x^2} + c.$$

10. $(\sin^{-1} x)^2$

Sol. Put $x = \sin\theta$. Differentiating both sides, $dx = \cos\theta\, d\theta$

$$\therefore \int (\sin^{-1} x)^2\, dx = \int \underset{\text{I}}{\theta^2}\, \underset{\text{II}}{\cos\theta}\, d\theta = \theta^2 \sin\theta - \int 2\theta \sin\theta\, d\theta$$

$$= \theta^2 \sin\theta - 2\int \underset{\text{I}}{\theta}\, \underset{\text{II}}{\sin\theta}\, d\theta$$

$= \theta^2 \sin\theta - 2\left[\theta(-\cos\theta) - \int 1.(-\cos\theta)\,d\theta\right]$

$= \theta^2 \sin\theta + 2\theta\cos\theta - 2\int \cos\theta\,d\theta = \theta^2\sin\theta + 2\theta\cos\theta - 2\sin\theta + c$

$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\,\sin^{-1}x - 2x + c.$

$\left(\because \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}\right)$

11. $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Sol. Let $I = \int \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}\,dx$ ...(*i*)

Put $\cos^{-1}x = \theta$. $\Rightarrow x = \cos\theta$

Therefore $\dfrac{dx}{d\theta} = -\sin\theta \Rightarrow dx = -\sin\theta\,d\theta$

$\therefore$ From (*i*), $I = \int \dfrac{(\cos\theta)\,\theta}{\sqrt{1-\cos^2\theta}}(-\sin\theta\,d\theta)$

$= -\int \dfrac{\theta\cos\theta\sin\theta}{\sin\theta}\,d\theta \quad (\because \sqrt{1-\cos^2\theta} = \sqrt{\sin^2\theta} = \sin\theta)$

$= -\int \underset{\text{I}}{\theta}\,\underset{\text{II}}{\cos\theta}\,d\theta$

Applying Product Rule: $\text{I}\int \text{II}\,d\theta - \int\left[\dfrac{d}{d\theta}(\text{I})\int \text{II}\,d\theta\right]d\theta$

$= -\left[\theta.\sin\theta - \int 1.\sin\theta\,d\theta\right] = -\theta\sin\theta + \int \sin\theta\,d\theta$

$= -\theta\sin\theta - \cos\theta + c = -\theta\sqrt{1-\cos^2\theta} - \cos\theta + c$

Putting $\theta = \cos^{-1}x$ and $\cos\theta = x$,

$= -(\cos^{-1}x)\sqrt{1-x^2} - x + c = -[\sqrt{1-x^2}\cos^{-1}x + x] + c.$

12. $x\sec^2 x$

Sol. $\int \underset{\text{I}}{x}\,\underset{\text{II}}{\sec^2 x}\,dx$

Applying Product Rule: $\text{I}\int \text{II}\,dx - \int\left[\dfrac{d}{dx}(\text{I})\int \text{II}\,dx\right]dx$

$$= x \int \sec^2 x \, dx - \int \left[\frac{d}{dx}(x) \int \sec^2 x \, dx \right] dx$$

$$= x \tan x - \int 1 . \tan x \, dx = x \tan x - \int \tan x \, dx$$

$$= x \tan x - (-\log|\cos x|) + c = x \tan x + \log|\cos x| + c.$$

13. $\tan^{-1} x$

Sol. Let $I = \int \tan^{-1} x \, dx = \int (\tan^{-1} x) . 1 \, dx$

$$= \tan^{-1} x . x - \int \frac{1}{1 + x^2} . x \, dx \quad = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1 + x^2} dx$$

$$= x \tan^{-1} x - \frac{1}{2} \log|(1 + x^2)| + c. \left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| \right]$$

$= x \tan^{-1} x - \frac{1}{2} \log(1 + x^2) + c$
$[\because \ 1 + x^2 \geq 1 > 0$ and hence $|1 + x^2| = 1 + x^2]$

14. $x (\log x)^2$

Sol. $\int x (\log x)^2 \, dx = \int \underset{\text{I}}{(\log x)^2} \cdot \underset{\text{II}}{x} \, dx$

Applying Product Rule: $\text{I} \int \text{II} \, dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$= (\log x)^2 \int x \, dx - \int \left[\frac{d}{dx} (\log x)^2 \int x \, dx \right] dx$

$= (\log x)^2 \frac{x^2}{2} - \int \frac{2 (\log x)}{x} \frac{x^2}{2} \, dx$

$\left[\because \frac{d}{dx} (\log x)^2 = 2(\log x)^1 \frac{d}{dx} (\log x) = 2 \log x \cdot \frac{1}{x} = \frac{2 \log x}{x} \right]$

$= \frac{x^2}{2} (\log x)^2 - \int \underset{\text{I}}{(\log x)} \underset{\text{II}}{x} \, dx$ $\left[\because \frac{x^2}{x} = \frac{x \cdot x}{x} = x \right]$

Again applying Product Rule: $\text{I} \int \text{II} \, dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$= \frac{x^2}{2} (\log x)^2 - \left[(\log x) \frac{x^2}{2} - \int \left(\frac{1}{x} \frac{x^2}{2} \right) dx \right] + c$

$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int x \, dx + c$

$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + c.$

15. $(x^2 + 1) \log x$

Sol. $\int (x^2 + 1) \log x \, dx = \int \underset{\text{I}}{(\log x)} \underset{\text{II}}{(x^2 + 1)} \, dx$

Applying Product Rule: $\text{I} \int \text{II} \, dx - \int \left[\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right] dx$

$= \log x \left(\frac{x^3}{3} + x \right) - \int \frac{1}{x} \left(\frac{x^3}{3} + x \right) dx$

$\left(\frac{x^3}{} + x \right) \quad \left(\frac{x^2}{} + \right)$

$$= \left(\frac{x^3}{3} + x\right) \log x - \int \left(\frac{x^2}{3} + 1\right) dx$$

$$= \left(\frac{x^3}{3} + x\right) \log x - \frac{1}{3} \int x^2 \, dx - \int 1 \, dx$$

$$= \left(\frac{x^3}{3} + x\right) \log x - \frac{1}{3} \frac{x^3}{3} - x + c = \left(\frac{x^3}{3} + x\right) \log x - \frac{x^3}{9} - x + c.$$

Integrate the functions in Exercises 16 to 22:

16. $e^x (\sin x + \cos x)$

Sol. Here $I = \int e^x (\sin x + \cos x)\, dx$

It is of the form $\int e^x [f(x) + f'(x)]\, dx$

Let us take $f(x) = \sin x$ so that $f'(x) = \cos x$

$$I = e^x f(x) + c = e^x \sin x + c. \quad \left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c\right]$$

17. $\dfrac{x e^x}{(1+x)^2}$

Sol. Here $I = \int \dfrac{xe^x}{(x+1)^2}\, dx = \int \dfrac{(x+1)-1}{(x+1)^2} e^x\, dx$

$$= \int e^x \left[\frac{x+1}{(x+1)^2} - \frac{1}{(x+1)^2}\right] dx = \int e^x \left[\frac{1}{x+1} + \frac{-1}{(x+1)^2}\right] dx$$

It is of the form $\int e^x [f(x) + f'(x)]\, dx$

Let us take $f(x) = \dfrac{1}{x+1}$ so that $f'(x) = \dfrac{d}{dx}[(x+1)^{-1}]$

$$= -(x+1)^{-2} = \frac{-1}{(x+1)^2}$$

$$\therefore\ I = e^x f(x) + c = \frac{e^x}{x+1} + c. \quad \left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + c\right]$$

18. $e^x \left(\dfrac{1+\sin x}{1+\cos x}\right)$

Sol. Here $I = \int e^x . \dfrac{1+\sin x}{1+\cos x}\, dx = \int e^x . \dfrac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\, dx$

$$= \int e^x . \left[\frac{1}{2\cos^2\frac{x}{2}} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right] dx = \int e^x \left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) dx$$

$$= \int e^x \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right) dx$$

It is of the form $\int e^x \, [f(x) + f'(x)] \, dx$

Let us take $f(x) = \tan \frac{x}{2}$ so that $f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$

$\therefore \; I = e^x f(x) + c = e^x \tan \frac{x}{2} + c.$

$\left[\because \int e^x (f(x) + f'(x)) \, dx = e^x f(x) + c\right]$

19. $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Sol. Let $I = \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right) dx$

It is of the form $\int e^x (f(x) + f'(x))\ dx$

Here $f(x) = \frac{1}{x} = x^{-1}$ and so $f'(x) = (-1)\, x^{-2} = \frac{-1}{x^2}$

$\therefore$ $I = e^x f(x) + c$ $\quad [\because \int e^x (f(x) + f'(x)]\, dx = e^x f(x) + c$

$= e^x \frac{1}{x} + c = \frac{e^x}{x} + c.$

20. $\frac{(x-3)\, e^x}{(x-1)^3}$

Sol. Here $I = \int \frac{(x-3)\, e^x}{(x-1)^3} dx = \int \frac{(x-1)-2}{(x-1)^3} e^x\, dx$

$= \int e^x \left[\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3}\right] dx = \int e^x \left[\frac{1}{(x-1)^2} + \frac{-2}{(x-1)^3}\right] dx$

It is of the form $\int e^x [f(x) + f'(x)]\ dx$

Let us take $f(x) = \frac{1}{(x-1)^2}$ so that $f'(x) = \frac{d}{dx}[(x-1)^{-2}]$

$= -2(x-1)^{-3} = \frac{-2}{(x-1)^3}$

$\therefore$ $I = e^x f(x) + c = \frac{e^x}{(x-1)^2} + c.$

$\left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x)\right]$

Note. Rule to evaluate $\int e^{ax} \sin bx\, dx$ or $\int e^{ax} \cos bx\, dx$

Let $I = \int e^{ax} \sin bx\, dx$ or $\int e^{ax} \cos bx\, dx$

I II I II

Integrate twice by product Rule and transpose term containing I from R.H.S. to L.H.S.

21. $e^{2x} \sin x$

Sol. Let $I = \int \underset{I}{e^{2x}} \underset{II}{\sin x}\, dx$ ...(*i*)

Applying Product Rule: $I \int II\, dx - \int \left[\frac{d}{dx}(I) \int II\, dx \right] dx$

$\Rightarrow \quad I = e^{2x}(-\cos x) - \int e^{2x} . 2 . (-\cos x)\, dx$

$\left[\because \frac{d}{dx} e^{2x} = e^{2x} \frac{d}{dx}(2x) = 2e^{2x} \right]$

$$\Rightarrow \quad I = -e^{2x}\cos x + 2\int \underset{\text{I}}{e^{2x}}\ \underset{\text{II}}{\cos x}\ dx$$

Again Applying Product Rule:

$$I = -e^{2x}\cos x + 2\left[e^{2x}\sin x - \int 2e^{2x}\sin x\,dx\right]$$

$$\Rightarrow \quad I = -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$$

$$\Rightarrow \quad I = e^{2x}(-\cos x + 2\sin x) - 4I \qquad \text{[By } (i)\text{]}$$

Transposing – 4I to L.H.S.; $5I = e^{2x}(2\sin x - \cos x)$

$$\therefore \quad I\left(=\int e^{2x}\sin x\,dx\right) = \frac{e^{2x}}{5}(2\sin x - \cos x) + c$$

Remark: The above question can also be done as:
Applying Product Rule: taking $\sin x$ as first function and e^{2x} as second function.

22. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol. Put $x = \tan\theta$. Differentiating both sides $dx = \sec^2\theta\,d\theta$.

$$\therefore \quad \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right).\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}(\sin 2\theta).\sec^2\theta\,d\theta = \int 2\theta\sec^2\theta\,d\theta$$

$$= 2\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sec^2\theta}\,d\theta$$

Applying product rule

$$= 2\left[\theta.\tan\theta - \int 1.\tan\theta\,d\theta\right] = 2\left[\theta\tan\theta - \int \tan\theta\,d\theta\right]$$

$$= 2[\theta\tan\theta - \log\sec\theta] + c$$

$$= 2[\tan^{-1}x.x - \log\sqrt{1+x^2}] + c$$

$$\left[\because \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+x^2}\right]$$

$$= 2\left[x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right] + c$$

$$= 2x\tan^{-1}x - \log(1+x^2) + c.$$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3}\,dx$ **equals**

(A) $\frac{1}{3}$ 2 (C) 1 e^{x^3} + C

(B) $\frac{1}{3} e^{x^2} + C$

$e^{x^3} + C$

$\frac{1}{2} e^{x^2} + C$

(D) $\frac{1}{\;}$

Sol. Let $I = \int x^2 e^{x^3} \, dx = \frac{1}{3} \int e^{(x^3)} (3x^2) \, dx \quad \left[\because \frac{d}{dx} x^3 = 3x^2 \right]$ ...(i)

Put $x^3 = t$. Therefore $3x^2 = \frac{dt}{dx}$. Therefore $3x^2 \, dx = dt$

$\therefore$ From (i), $I = \frac{1}{3}\int e^t\, dt = \frac{1}{3} e^t + C$

Putting $t = x^3$, $= \frac{1}{3} e^{x^3} + C$

$\therefore$ Option (B) is the correct answer.

24. $\int e^x$ **sec** x **(1 + tan** x**)** dx **equals**

(A) $e^x \cos x + C$ **(B)** $e^x \sec x + C$

(C) $e^x \sin x + C$ **(D)** $e^x \tan x + C$

Sol. Let $I = \int e^x \sec x\,(1 + \tan x)\, dx = \int e^x (\sec x + \sec x \tan x)\, dx$

It is of the form $\int e^x (f(x) + f'(x))\, dx$

Here $f(x) = \sec x$ and so $f'(x) = \sec x \tan x$

$\therefore$ $I = e^x f(x) + C$ $\left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + C\right]$

$= e^x \sec x + C$

$\therefore$ Option (B) is the correct answer.

Exercise 7.7

I. Rule to evaluate $\int \sqrt{\text{Pure Quadratic}}\ dx$, *i.e.*,

$$\int \sqrt{ax^2 + b}\ dx.$$

Apply directly one of these formulae according to form of integrand:

1. $\int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

2. $\int \sqrt{x^2 + a^2}\ dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right|.$

3. $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|.$

***II.* Rule to evaluate $\int \sqrt{\text{Quadratic}}\ dx$, *i.e.*, $\int \sqrt{ax^2 + bx + c}\ dx$**

Step I. Make coefficient of x^2 unity by taking $|a|$ common.

Now complete the squares by adding and subtracting $\left(\frac{1}{2}\text{Coefficient of } x\right)^2$.

Now applying one of the above three formulae (according to the form of the integrand) will give value of required integral.

Integrate the functions in Exercises 1 to 9:

1. $\sqrt{4 - x^2}$

Sol. $\int \sqrt{4 - x^2}\ dx = \int \sqrt{2^2 - x^2}\ dx$

$$= \frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} + c$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + c$$

2. $\sqrt{1 - 4x^2}$

Sol. $\int \sqrt{1 - 4x^2}\, dx = \int \sqrt{1^2 - (2x)^2}\, dx$

$$= \frac{\frac{(2x)}{2}\sqrt{1^2 - (2x)^2} + \frac{1^2}{2}\sin^{-1}\left(\frac{2x}{1}\right)}{2} + c \quad [2 \rightarrow \text{Coefficient of } x \text{ in } 2x]$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{2}\left[x\sqrt{1 - 4x^2} + \frac{1}{2}\sin^{-1}\frac{2x}{1}\right] + c = \frac{x}{2}\sqrt{1 - 4x^2} + \frac{1}{4}\sin^{-1} 2x + c.$$

3. $\sqrt{x^2 + 4x + 6}$

Sol. $\int \sqrt{x^2 + 4x + 6}\, dx$

Coefficient of x^2 is unity. So let us complete squares by adding and subtracting $\left(\frac{1}{2}\text{Coefficient of } x\right)^2 = 2^2$

$$= \int \sqrt{x^2 + 4x + 4 + 6 - 4}\, dx = \int \sqrt{(x+2)^2 + 2}\, dx$$

$$= \int \sqrt{(x+2)^2 + (\sqrt{2})^2}\, dx = \left(\frac{x+2}{2}\right)\sqrt{(x+2)^2 + (\sqrt{2})^2}$$

$$+ \frac{(\sqrt{2})^2}{2}\log\left|x + 2 + \sqrt{(x+2)^2 + (\sqrt{2})^2}\right| + c$$

$$\left[\because \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right|\right]$$

$$= \frac{(x+2)}{2}\sqrt{x^2 + 4 + 4x + 2}$$

$$+ \frac{2}{2}\log\left|x + 2 + \sqrt{x^2 + 4 + 4x + 2}\right| + c$$

Sol. $(x +$

4.

$$\frac{2}{)} \quad = \quad + \log \left| x + 2 + \sqrt{x^2 + 4x + 6} \right| + c. \qquad \sqrt{x^2 + 4x + 6}$$

2

$$\int \sqrt{\boldsymbol{x^2 + 4x + 1}} \, dx = \int \sqrt{x^2 + 4x + 1} \, dx \qquad \sqrt{x^2 + 4x + 2^2 + 1 - 4}$$

$$\left(\text{We have added and subtracted } \left(\frac{1}{2} \text{ coefficient of } x \right)^2 = 2^2 \right)$$

$= \int \sqrt{(x+2)^2 - 3}\, dx = \int \sqrt{(x+2)^2 - (\sqrt{3})^2}\, dx$

$= \left(\frac{x+2}{2}\right)\sqrt{(x+2)^2 - (\sqrt{3})^2} - \frac{(\sqrt{3})^2}{2} \log\left|x + 2 + \sqrt{(x+2)^2 - (\sqrt{3})^2}\right| + c$

$\left[\because \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right|\right]$

$= \left(\frac{x+2}{2}\right)\sqrt{x^2 + 4x + 1} - \frac{3}{2} \log\left|x + 2 + \sqrt{x^2 + 4x + 1}\right| + c$

$[\because (x+2)^2 - (\sqrt{3})^2 = x^2 + 4x + 4 - 3 = x^2 + 4x + 1]$

5. $\int \sqrt{1 - 4x - x^2}$

Sol. $\int \sqrt{1 - 4x - x^2}\, dx = \int \sqrt{-x^2 - 4x + 1}\, dx$

Making coefficient of x^2 unity

$= \int \sqrt{-(x^2 + 4x - 1)}\, dx$

(**Note.** You can't take this (–) sign out of this bracket because square root of –1 is imaginary)

$= \int \sqrt{-(x^2 + 4x + 2^2 - 4 - 1)}\, dx = \int \sqrt{-[(x+2)^2 - 5]}\, dx$

$= \int \sqrt{5 - (x+2)^2}\, dx = \int \sqrt{(\sqrt{5})^2 - (x+2)^2}\, dx$

$= \frac{x+2}{2}\sqrt{(\sqrt{5})^2 - (x+2)^2} + \frac{(\sqrt{5})^2}{2} \sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$

$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}\right]$

$= \frac{x+2}{2}\sqrt{1 - 4x - x^2} + \frac{5}{2} \sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$

$[\because (\sqrt{5})^2 - (x+2)^2 = 5 - (x^2 + 4 + 4x)$
$= 5 - x^2 - 4 - 4x = 1 - 4x - x^2]$

6. $\sqrt{x^2 + 4x - 5}$

$\sqrt{x^2 + 4x - 5}$ $\qquad \sqrt{x^2 + 4x + 2^2 - 4 - 5}$

Sol. $\int \quad dx = \int \quad dx$

$$= \int \sqrt{(x+2)^2 - 9}\, dx = \int \sqrt{(x+2)^2 - 3^2}\, dx$$

$$= \left(\frac{x+2}{2}\right)\sqrt{(x+2)^2 - 3^2} - \frac{3^2}{2}\log\left|x + 2 + \sqrt{(x+2)^2 - 3^2}\right| + c$$

$$\left[\because \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|\right]$$

$$= \left(\frac{x+2}{2}\right)\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|x+2+\sqrt{x^2+4x-5}\right| + c$$

$[\because (x+2)^2 - 3^2 = x^2+4x+4-9 = x^2+4x-5]$

7. $\sqrt{1+3x-x^2}$

Sol. $\int \sqrt{1+3x-x^2}\,dx = \int \sqrt{-x^2+3x+1}\,dx$

$$= \int \sqrt{-(x^2-3x-1)}\,dx$$

$$= \int \sqrt{-\left[x^2-3x+\left(\frac{3}{2}\right)^2-\frac{9}{4}-1\right]}\,dx = \int \sqrt{-\left[\left(x-\frac{3}{2}\right)^2-\frac{13}{4}\right]}\,dx$$

$$= \int \sqrt{\frac{13}{4}-\left(x-\frac{3}{2}\right)^2}\,dx = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

$$= \frac{x-\frac{3}{2}}{2}\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2} + \frac{\left(\frac{\sqrt{13}}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + c$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left(\frac{2x-3}{4}\right)\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + c$$

$$\left[\because \left(\frac{\sqrt{13}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 = \frac{13}{4} - \left(x^2+\frac{9}{4}-3x\right) = \frac{13}{4} - x^2 - \frac{9}{4} + 3x = 1+3x-x^2\right]$$

8. $\sqrt{x^2+3x}$

Sol. $\int \sqrt{x^2+3x}\,dx = \int \sqrt{\ldots}$

4

$\left(\frac{3}{2}\right)^2$

4

$\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}$

$\sqrt{x^2+4x-5}$

$$= \int \frac{dx}{\sqrt{x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}} \qquad \sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}$$

$$dx$$

$$= \frac{x+\frac{3}{2}}{2} \qquad - \frac{\left(\frac{3}{2}\right)^2}{2}\log\left|x+\frac{3}{2}+\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\right|+c$$

$$\left[\because \int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \frac{2x+3}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x}\right| + c$$

$$\left[\because \left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = x^2+3x+\frac{9}{4}-\frac{9}{4} = x^2+3x\right]$$

9. $\sqrt{1+\frac{x^2}{9}}$

Sol. $\int \sqrt{1+\frac{x^2}{9}}\,dx = \int \sqrt{\frac{9+x^2}{3^2}}\,dx = \int \frac{\sqrt{x^2+3^2}}{3}\,dx = \frac{1}{3}\int \sqrt{x^2+3^2}\,dx$

$$= \frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+3^2}+\frac{3^2}{2}\log\left|x+\sqrt{x^2+3^2}\right|\right]+c$$

$$\left[\because \int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= \frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+c.$$

Choose the correct answer in Exercises 10 to 11:

10. $\int \sqrt{1+x^2}\,dx$ **is equal to**

(A) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right|+C$

(B) $\frac{2}{3}(1+x^2)^{3/2}+C$

(C) $\frac{2}{3}x(1+x^2)^{3/2}+C$

(D) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C.$

Sol. $\int \sqrt{1+x^2}\,dx = \int \sqrt{x^2+1^2}\,dx$

$$= \frac{x}{2}\sqrt{x^2+1^2}+\frac{1^2}{2}\log\left|x+\sqrt{x^2+1^2}\right|+C$$

$$\left[\because \int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= \frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\ldots$$

$= \quad 2 \qquad 2 \log \left| x + \sqrt{x^2+1} \right| + C.$

11. $\int \sqrt{x^2 - 8x + 7}\, dx$ **is equal to**

(A) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} + 9\log\left| x - 4 + \sqrt{x^2 - 8x + 7} \right| + C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2 - 8x + 7} + 9\log\left| x + 4 + \sqrt{x^2 - 8x + 7} \right| + C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} - 3\sqrt{2}\log\left| x - 4 + \sqrt{x^2 - 8x + 7} \right| + C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2 - 8x + 7} - \frac{9}{2}\log\left| x - 4 + \sqrt{x^2 - 8x + 7} \right| + C.$

Sol. $\int \sqrt{x^2 - 8x + 7}\, dx = \int \sqrt{x^2 - 8x + 4^2 - 16 + 7}\, dx$

$= \int \sqrt{(x-4)^2 - 9}\, dx = \int \sqrt{(x-4)^2 - 3^2}\, dx$

$= \left(\frac{x-4}{2}\right)\sqrt{(x-4)^2 - 3^2} - \frac{3^2}{2}\log\left|x - 4 + \sqrt{(x-4)^2 - 3^2}\right| + C$

$\left[\because \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|\right]$

$= \left(\frac{x-4}{2}\right)\sqrt{x^2 - 8x + 7} - \frac{9}{2}\log\left|x - 4 + \sqrt{x^2 - 8x + 7}\right| + C.$

$[\because (x-4)^2 - 3^2 = x^2 - 8x + 16 - 9 = x^2 - 8x + 7]$

Exercise 7.8

Definition of definite integral as the limit of a sum:

$$\int_a^b f(x)\ dx = \underset{\substack{h\to 0\\ n\to\infty}}{\text{Lt}}\ h[f(a) + f(a+h) + f(a+2h) + \dots\dots\dots + f(a+(n-1)\ h)]$$

where $nh = b - a$

Note. The series within brackets represents the sum of n terms.

Evaluate the following definite integrals as limit of sums:

1. $\int_a^b x\ dx$

Sol. Step I. Comparing $\int_a^b x\ \ dx$ with $\int_a^b f(x)\ \ dx$ we have

$$a = a,\ b = b \text{ and } f(x) = x \quad \dots(i)$$

$$\therefore \quad nh = b - a = b - a$$

Step II. Putting $x = a, a + h, a + 2h, \dots\dots\dots, a + (n - 1)\ h$ in (i), we have $f(a) = a$, $f(a + h) = a + h$,

$f(a + 2h) = a + 2h, \dots\dots\dots, f(a + (n - 1)h) = a + (n - 1)h$

Step III. Putting these values in

$$\int_a^b f(x)\ dx = \underset{\substack{h\to 0\\ n\to\infty}}{\text{Lt}}\ h\ [f(a) + f(a+h) + f(a+2h) + \dots\dots\dots + f(a+(n-1)h)]$$

where $nh = b - a$, we have

$$\int_a^b x\ dx = \underset{h\to 0}{\text{Lt}}\ h\ [a + (a+h) + (a+2h) + \dots\dots\dots + (a+(n-1)h)]$$

where $nh = b - a$

$$= \underset{\substack{h\to 0\\ n\to\infty}}{\text{Lt}}\ h[na + h(1 + 2 + 3 + \dots\dots\dots + (n-1)]$$

$$= \lim_{\substack{h \to 0 \\ n \to \infty}} \left[anh + hh \frac{n(n-1)}{2} \right] \left[\because 1 + 2 + 3 + \dots + (n-1) = \frac{n(n-1)}{2} \right]$$

$$= \lim_{\substack{h \to 0 \\ n \to \infty}} \left[anh + \frac{nh(nh - h)}{2} \right].$$

Step IV. Putting $nh = b - a$,

$$= \lim_{h \to 0} \left[a(b-a) + \frac{(b-a)(b-a-h)}{2} \right].$$

Step V. Taking Limits as $h \to 0$ (*i.e.,* putting $h = 0$ here)

$$= a(b-a) + \frac{(b-a)(b-a)}{2}$$

$$= (b-a)\left[a + \frac{b-a}{2} \right] = (b-a)\left[\frac{2a + b - a}{2} \right]$$

$$= \frac{(b-a)(b+a)}{2} = \frac{b^2 - a^2}{2}.$$

2. $\int_0^5 (x+1)\,dx$

Sol. Step I. Comparing $\int_0^5 (x+1)\,dx$ with $\int_a^b f(x)\,dx$, we have

$a = 0,\ b = 5$ and $f(x) = x + 1$ ...(*i*)

$\therefore\quad nh = b - a = 5 - 0 = 5.$

Step II. Putting $x = a, a + h, a + 2h, \dots, a + (n-1)h$ in (*i*), we have

$f(a) = f(0) = 0 + 1 = 1,\ f(a + h) = f(h) = h + 1,$

$f(a + 2h) = f(2h) = 2h + 1, \dots,$

$f(a + (n-1)h) = f((n-1)h) = (n-1)h + 1.$

Step III. Putting these values in

$$\int_a^b f(x)\,dx = \lim_{\substack{h \to 0 \\ n \to \infty}} h[f(a) + f(a+h) + f(a+2h) + \dots + f(a + (n-1)h)],$$ we have

$$\int_0^5 (x+1)\,dx = \lim_{\substack{h \to 0 \\ n \to \infty}} h[1 + (h+1) + (2h+1) + \dots + [(n-1)h + 1]]$$

$$= \lim h[n + h(1 + 2 + \dots + (n-1)] = \lim \left[nh + hh \frac{n(n-1)}{\dots} \right]$$

$\substack{h \to 0 \\ n \to \infty}$ $\qquad\qquad$ $\substack{h \to 0 \\ n \to \infty}$ $\quad 2$

$$\left[\because\ 1 + 2 + 3 + \ldots\ldots + (n-1) = \frac{n(n-1)}{2}\right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[nh + \frac{(nh)(nh-h)}{2}\right].$$

Step IV. Putting $nh = 5$, $= \underset{h \to 0}{\text{Lt}} \left[5 + \frac{5(5-h)}{2} \right]$.

Step V. Taking limits as $h \to 0$ (*i.e.*, putting $h = 0$ here)

$$= 5 + \frac{5(5-0)}{2} = 5 + \frac{25}{2} = \frac{10+25}{2} = \frac{35}{2}.$$

3. $\int_2^3 x^2 \, dx$

Sol. Step I. Comparing $\int_2^3 x^2 \, dx$ with $\int_a^b f(x)$, we have

$a = 2,\ b = 3$ and $f(x) = x^2$ ...(*i*)

$\therefore\ nh = b - a = 3 - 2 = 1.$

Step II. Putting $x = a,\ a + h,\ a + 2h, \ldots\ldots,\ a + (n-1)h$ in (*i*), we have

$f(a) = f(2) = 2^2 = 4$

$f(a + h) = f(2 + h) = (2 + h)^2 = 4 + 4h + h^2$

$f(a + 2h) = f(2 + 2h) = (2 + 2h)^2 = 4 + 8h + 2^2h^2$

$f(a + (n-1)h) = f(2 + (n-1)h) = (2 + (n-1)h)^2$

$= 4 + 4(n-1)h + (n-1)^2h^2.$

Step III. Putting these values in

$$\int_a^b f(x)\,dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a + (n-1)h)]$$

where $nh = 1$, we have

$$\int_2^3 x^2 \, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[4 + (4 + 4h + h^2) + (4 + 8h + 2^2h^2) + \ldots\ldots + (4 + 4(n-1)h + (n-1)^2h^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[4n + 4h(1 + 2 + \ldots\ldots + (n-1)) + h^2(1^2 + 2^2 + \ldots\ldots + (n-1)^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[4nh + 4hh\frac{n(n-1)}{2} + hhh\frac{n(n-1)(2n-1)}{6} \right]$$

$$\left[\because\ 1 + 2 + \ldots + (n-1) = \frac{n(n-1)}{2} \text{ and } 1^2 + 2^2 + \ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} \right]$$

$$= \text{Lt} \left[4nh + 4nh\frac{(nh-h)}{\ldots} + \frac{nh(nh-h)(2nh-h)}{\ldots} \right].$$

$$\lim_{\substack{h \to 0 \\ n \to \infty}} \left[\qquad 2 \qquad\qquad 6 \qquad \right]$$

Step IV. Putting $nh = 1$;

$$= \lim_{h \to 0} \left[4 + 2(1-h) + 1\frac{(1-h)(2-h)}{6} \right].$$

Step V. Taking limits as $h \to 0$ (*i.e.,* putting $h = 0$ here)

$$= 4 + 2(1 - 0) + \frac{1(2)}{6} = 6 + \frac{1}{3} = \frac{19}{3}.$$

4. $\int_1^4 (x^2 - x)\, dx$

Sol. Step I. Comparing $\int_1^4 (x^2 - x)\, dx$ with $\int_a^b f(x)\, dx$,

we have

$$a = 1,\ b = 4,\ f(x) = x^2 - x \qquad ...(i)$$

$$\therefore \quad nh = b - a = 4 - 1 = 3.$$

Step II. Putting $x = a, a + h, a + 2h, ... a + (n - 1)h$ in (i),

$$f(a) = f(1) = 1^2 - 1 = 1 - 1 = 0$$

$$f(a + h) = f(1 + h) = (1 + h)^2 - (1 + h)$$

$$= 1 + h^2 + 2h - 1 - h = h + h^2$$

$$f(a + 2h) = f(1 + 2h) = (1 + 2h)^2 - (1 + 2h)$$

$$= 1 + 4h^2 + 4h - 1 - 2h$$

$$= 2h + 4h^2$$

$$f(a + (n - 1)h) = (1 + (n - 1)h)^2 - (1 + (n - 1)h)$$

$$= 1 + (n - 1)^2 h^2 + 2(n - 1)h - 1 - (n - 1)h$$

$$= (n - 1)h + (n - 1)^2 h^2.$$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[f(a) + f(a + h) + f(a + 2h) + ... + f(a + (n - 1)h)]$$

we have

$$\int_1^4 (x^2 - x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[0 + h + h^2 + 2h + 4h^2 + ... + (n - 1)h + (n - 1)^2 h^2)$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}}\ h[h(1 + 2 + ... + (n - 1)) + h^2(1^2 + 2^2 + ... + (n - 1)^2)]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[h \cdot h \cdot \frac{n(n-1)}{2} + h \cdot h \cdot h\, \frac{n(n-1)(2n-1)}{6} \right]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} \left[nh\, \frac{(nh - h)}{2} + \frac{(nh)(nh - h)(2nh - h)}{6} \right].$$

Step IV. Putting $nh = 3$

$$= \text{Lt} \left[\frac{3(3-h)}{} + \frac{3(3-h)(6-h)}{} \right]$$

$h \to 0$ $\lfloor$ 2 6 $\rfloor$

Step V. Taking limits as $h \to 0$ (Putting $h = 0$ here)

$$= \frac{3(3-0)}{2} + \frac{3(3-0)(6-0)}{6} = \frac{9}{2} + 9 = \frac{27}{2}.$$

5. $\int_{-1}^{1} e^x \, dx$

Sol. Step I. Comparing $\int_{-1}^{1} e^x \, dx$ with $\int_a^b f(x) \, dx$, we have

$a = -1,\ b = 1$ and $f(x) = e^x$ ...(i)

$\therefore \ nh = b - a = 1 - (-1) = 2.$

Step II. Putting $x = a, a + h, a + 2h, \ldots\ldots, a + (n - 1)h$ in (i), we have

$f(a) = f(-1) = e^{-1}$

$f(a + h) = f(-1 + h) = e^{-1+h} = e^{-1} . e^h$

$f(a + 2h) = f(-1 + 2h) = e^{-1+2h} = e^{-1} . e^{2h}$

$f(a + (n - 1)h) = f(-1 + (n - 1)h) = e^{-1+(n-1)h} = e^{-1} e^{(n-1)h}.$

Step III. Putting these values in

$$\int_a^b f(x)\, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1)h)],$$

we have

$$\int_{-1}^{1} e^x \, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h[e^{-1} + e^{-1} e^h + e^{-1} e^{2h} + \ldots\ldots + e^{-1} e^{(n-1)h}]$$

$$= \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h\, e^{-1} \frac{[(e^h)^n - 1]}{e^h - 1}$$ [$\because$ The series within brackets is a G.P. series with First term A = e^{-1} and common ratio R = e^h, Number of terms is n and S_n of G.P. = $A \left[\frac{(R^n - 1)}{R - 1}\right]$.

$$= \int_{-1}^{1} e^x \, dx = \underset{\substack{h \to 0 \\ n \to \infty}}{\text{Lt}} h\, e^{-1} \frac{(e^{nh} - 1)}{e^h - 1}.$$

Step IV. Putting $nh = 2$, $= \underset{h \to 0}{\text{Lt}}\ h\, e^{-1} \dfrac{(e^2 - 1)}{e^h - 1}$

$$= e^{-1}(e^2 - 1) \underset{h \to 0}{\text{Lt}} \frac{h}{e^h - 1} = e^{-1}(e^2 - 1) \times 1 \left[\because \underset{x \to 0}{\text{Lt}} \frac{x}{e^x - 1} = 1\right]$$

$= e^{-1+2} - e^{-1} = e^{1} - e^{-1} = e - e^{-1}$.

6. $\int_0^4 (x + e^{2x})\, dx$

Sol. Step I. Comparing $\int_0^4 (x + e^{2x})\, dx$ with $\int_a^b f(x)\, dx$, we have

$a = 0,\ b = 4$ and $f(x) = x + e^{2x}$ ...(i)

$\therefore \quad nh = b - a = 4 - 0 = 4.$

Step II. Putting $x = a, a + h, a + 2h, ..., a + (n-1)h$ in (i), we have

$f(a) = f(0) = 0 + e^0 = 1$

$f(a + h) = f(h) = h + e^{2h}$

$f(a + 2h) = f(2h) = 2h + e^{4h}$

$f(a + (n - 1)h) = f((n - 1)h) = (n - 1)h + e^{2(n-1)h}$.

Step III. Putting these values in

$$\int_a^b f(x)\,dx = \lim_{\substack{h\to 0\\ n\to\infty}} h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1)h)],$$

we have

$$\int_0^4 (x + e^{2x})\,dx = \lim_{\substack{h\to 0\\ n\to\infty}} h[1 + (h + e^{2h}) + (2h + e^{4h}) + \ldots\ldots ((n - 1)h + e^{2(n-1)h})]$$

(G.P. series : A = 1, R = e^{2h}, $n = n$)

$$= \lim_{\substack{h\to 0\\ n\to\infty}} h[(h + 2h + .. + (n - 1)h) + (1 + e^{2h} + e^{4h} + \ldots + e^{2(n-1)h})]$$

$$= \lim_{\substack{h\to 0\\ n\to\infty}} h\left[h(1 + 2 + \ldots\ldots + (n - 1)) + A\left(\frac{R^n - 1}{R - 1}\right)\right]$$

$$= \lim_{\substack{h\to 0\\ n\to\infty}} h\left[h\frac{n(n-1)}{2} + \frac{1((e^{2h})^n - 1)}{e^{2h} - 1}\right]$$

$$= \lim_{\substack{h\to 0\\ n\to\infty}} \left[\frac{nh(nh - h)}{2} + \frac{h(e^{2nh} - 1)}{e^{2h} - 1}\right].$$

Step IV. Putting $nh = 4$, $= \lim_{h\to 0}\left[\frac{4(4 - h)}{2} + \frac{h(e^8 - 1)}{e^{2h} - 1}\right]$.

Step V. Taking limits as $h \to 0$

$$= \frac{4(4 - 0)}{2} + (e^8 - 1)\lim_{h\to 0}\frac{h}{e^{2h} - 1} = 8 + (e^8 - 1)\frac{1}{2}\lim_{h\to 0}\frac{2h}{e^{2h} - 1}$$

$$= 8 + \frac{(e^8 - 1)}{2}.\left[\because \lim_{h\to 0}\frac{e^{2h}}{e^{2h} - 1}\left(\Rightarrow \lim_{x\to 0}\frac{x}{e^x - 1}\right) = 1\right]$$

Exercise 7.9

Evaluate the definite integrals in Exercises 1 to 11:

Result. If $\int f(x)\ dx = \phi(x)$, then $\int_a^b f(x)\ dx = \phi(b) - \phi(a)$ ...(*i*)

(This is known as **Second Fundamental Theorem**).

1. $\int_{-1}^{1} (x+1)\ dx$

Sol. $\int_{-1}^{1} (x+1)\, dx = \left(\frac{x^2}{2} + x\right)_{-1}^{1} = \phi(b) - \phi(a)$

(By Second Fundamental Theorem given in Eqn. (*i*) page 496)

$= \left(\frac{1^2}{2} + 1\right) - \left(\frac{(-1)^2}{2} - 1\right) = \frac{1}{2} + 1 - \left(\frac{1}{2} - 1\right)$

$= \frac{1}{2} + 1 - \frac{1}{2} + 1 = 2.$

Remark. [Constant c will never occur in the value of a definite integral because c in the value of $\phi(b)$ gets cancelled with c in $\phi(a)$ when we subtract them to get $\phi(b) - \phi(a)$].

2. $\int_2^3 \frac{1}{x}\, dx$

Sol. $\int_2^3 \frac{1}{x}\, dx = \left(\log|x|\right)_2^3 = \phi(b) - \phi(a) = \log|3| - \log|2|$

$= \log 3 - \log 2 = \log \frac{3}{2}.$ [........ $|x| = x$ if $x \geq 0$]

3. $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx$

Sol. $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx = \left(4\frac{x^4}{4} - 5\frac{x^3}{3} + 6\frac{x^2}{2} + 9x\right)_1^2$

$= \left(x^4 - \frac{5}{3}x^3 + 3x^2 + 9x\right)_1^2$

$= \left[2^4 - \frac{5}{3}(2)^3 + 3(2)^2 + 9(2)\right] - \left[1 - \frac{5}{3} + 3 + 9\right]$

$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(13 - \frac{5}{3}\right)$

$= 46 - \frac{40}{3} - \left(13 - \frac{5}{3}\right) = 46 - \frac{40}{3} - 13 + \frac{5}{3}$

$= 33 - \frac{40}{3} + \frac{5}{3} = \frac{99 - 40 + 5}{3} = \frac{104 - 40}{3} = \frac{64}{3}.$

4. $\int_0^{\frac{\pi}{4}} \sin 2x\, dx$

$\left(\frac{-\cos 2x}{2}\right)^{\frac{\pi}{4}}$ $-\cos\frac{\pi}{2}$ $\left(\frac{-\cos 0}{2}\right)$

Sol. $\int_0^{4} \sin 2x \; dx = \left[\frac{\quad}{2} \right]_0^{4} = \frac{\quad 2}{2} - \left(\frac{\quad}{2} \right)$

$$= \frac{-0}{2} - \left(\frac{-1}{2} \right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

5. $\int_0^{\frac{\pi}{2}} \cos 2x \, dx$

Sol. $\int_0^{\frac{\pi}{2}} \cos 2x \, dx = \left(\frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}} = \frac{\sin \pi}{2} - \frac{\sin 0}{2}$

$= \frac{0}{2} - \frac{0}{2} = 0$

[∵ $\sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0 = 0$]

6. $\int_4^5 e^x \, dx$

Sol. $\int_4^5 e^x \, dx = \left(e^x\right)_4^5 = e^5 - e^4 = e^4 (e - 1).$

7. $\int_0^{\frac{\pi}{4}} \tan x \, dx$

Sol. $\int_0^{\frac{\pi}{4}} \tan x \, dx = \left(\log |\sec x|\right)_0^{\frac{\pi}{4}}$

$= \log \left|\sec \frac{\pi}{4}\right| - \log |\sec 0| = \log |\sqrt{2}| - \log |1|$

$= \log \sqrt{2} - \log 1 = \log 2^{1/2} - 0 = \frac{1}{2} \log 2.$

8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

Sol. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx = \left(\log |\operatorname{cosec} x - \cot x|\right)_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \log \left|\operatorname{cosec} \frac{\pi}{4} - \cot \frac{\pi}{4}\right| - \log \left|\operatorname{cosec} \frac{\pi}{6} - \cot \frac{\pi}{6}\right|$

$= \log |\sqrt{2} - 1| - \log |2 - \sqrt{3}|$

$= \log (\sqrt{2} - 1) - \log (2 - \sqrt{3})$ [∵ $|x| = x$ if $x \geq 0$]

$= \log \left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right)$

9. $\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

Sol. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left(\sin^{-1} x\right)_0^1 \quad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}\right]$

$= \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}. \quad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \sin 0 = 0\right]$

10. $\int_0^1 \frac{dx}{1+x^2}$

Sol. $\int_0^1 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^1 \qquad \left[\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$

$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$

11. $\int_2^3 \frac{dx}{x^2 - 1}$

Sol. $\int_2^3 \frac{1}{x^2-1} dx = \int_2^3 \frac{1}{x^2-1^2} dx$

$= \left(\frac{1}{2(1)} \log\left|\frac{x-1}{x+1}\right|\right)_2^3 \left[\because \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

$= \frac{1}{2}\log\left|\frac{3-1}{3+1}\right| - \frac{1}{2}\log\left|\frac{2-1}{2+1}\right| = \frac{1}{2}\log\left|\frac{1}{2}\right| - \frac{1}{2}\log\left|\frac{1}{3}\right|$

$= \frac{1}{2}\left(\log\frac{1}{2} - \log\frac{1}{3}\right) \qquad [\because |x| = x \text{ if } x \geq 0]$

$= \frac{1}{2}\left[\log\left(\frac{\frac{1}{2}}{\frac{1}{3}}\right)\right] = \frac{1}{2}\log\frac{3}{2}.$

Evaluate the definite integrals in Exercises 12 to 20:

12. $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

Sol. $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2x) \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x) \, dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - \left(0 + \frac{1}{2}\sin 0\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0\right]$

$= \frac{\pi}{4}$. [∵ $\sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0 = 0$]

13. $\int_2^3 \frac{x\,dx}{x^2+1}$

Sol. $\int_2^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_2^3 \frac{2x}{x^2+1}\,dx$

$= \frac{1}{2}\left(\log|x^2+1|\right)_2^3$. $\left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)|\right]$

(Here $f(x) = x^2 + 1$ and $f'(x) = 2x$)

$$= \frac{1}{2}(\log|10| - \log|5|) = \frac{1}{2}(\log 10 - \log 5)$$

$$= \frac{1}{2}\log\frac{10}{5} = \frac{1}{2}\log 2.$$

14. $\int_0^1 \frac{2x+3}{5x^2+1}\, dx$

Sol. $\int_0^1 \frac{2x+3}{5x^2+1}\, dx = \int_0^1 \left(\frac{2x}{5x^2+1} + \frac{3}{5x^2+1}\right) dx$

$$= \int_0^1 \frac{2x}{5x^2+1}\, dx + 3\int_0^1 \frac{dx}{5x^2+1}$$

$$= \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}\, dx + 3\int_0^1 \frac{dx}{(\sqrt{5}x)^2+1^2}$$

$$= \frac{1}{5}\left(\log|5x^2+1|\right)_0^1 + 3\cdot\frac{1}{\sqrt{5}}\left(\tan^{-1}\left(\frac{\sqrt{5}x}{1}\right)\right)_0^1 \qquad \sqrt{5} \to \text{Coefficient of } x$$

$$\left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| \text{ and } \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{5}(\log 6 - \log 1) + \frac{3}{\sqrt{5}}(\tan^{-1}\sqrt{5} - \tan^{-1} 0)$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}.$$

15. $\int_0^1 x\, e^{x^2}\, dx$

Sol. To evaluate $\int_0^1 x\, e^{x^2}\, dx$

Let us first evaluate $\int x\, e^{x^2}\, dx$

$$= \frac{1}{2}\int e^{x^2}(2x\, dx) \qquad \ldots(i)$$

Put $x^2 = t$. Therefore $2x = \frac{dt}{dx}$ $\therefore$ $2x\, dx = dt$

$\therefore$ From (*i*), $\int x\, e^{x^2}\, dx = \frac{1}{2}\int e^t\, dt = \frac{1}{2} e^t$

Putting $t = x^2$, $= \frac{1}{2} e^{x^2}$ ...(*ii*)

$\therefore$ The given integral $\int_0^1 x\, e^{x^2}\, dx = \frac{1}{2}\left(e^{x^2}\right)_0^1$ [By (*ii*)]

$$= \frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e - 1).$$

Note. Please note that limits 0 and 1 specified in the given integral are limits for x.
Therefore after substituting $x^2 = t$ and evaluating the integral, we must put back $t = x^2$ and only then use $\int_a^b f(x)\ dx = \phi(b) - \phi(a)$.

Remark. In the next Exercise 7.10 we shall also learn to change the limits of integration from values of x to values of t and thenwe may use our discretion even here also.

16. $\int_1^2 \frac{5x^2}{x^2+4x+3}\ dx$

Sol. $\int_1^2 \frac{5x^2}{x^2+4x+3}\ dx = \int_1^2 \frac{5x^2}{(x+1)(x+3)}$ ...(*i*)

$[\because\ x^2 + 4x + 3 = x^2 + 3x + x + 3$
$= x(x + 3) + 1(x + 3) = (x + 1)(x + 3)]$

The integrand $\frac{5x^2}{(x+1)(x+3)}$ is a rational function and degree of numerator = degree of denominator.
So let us apply long division.

$(x + 1)(x + 3) = x^2 + 4x + 3\ )\ 5x^2\ (5$
$5x^2 + 20x + 15$
$-\quad -\quad -$
$-20x - 15$

$\therefore \frac{5x^2}{(x+1)(x+3)} = 5 + \frac{(-20x-15)}{(x+1)(x+3)}$

Putting this value in (*i*),

$$\int_1^2 \frac{5x^2}{x^2+4x+3}\ dx = \int_1^2 \left(5 + \frac{(-20x-15)}{(x+1)(x+3)}\right) dx$$

$$= \int_1^2 5\ dx + \int_1^2 \frac{-20x-15}{(x+1)(x+3)}\ dx = 5\,(x)_1^2 + \mathbf{I}$$

$= 5(2 - 1) + \mathbf{I} = 5 + \mathbf{I}$ ...(*ii*)

where I = $\int_1^2 \frac{-20x-15}{(x+1)(x+3)}\ dx$

Let integrand of I = $\frac{-20x-15}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$ ...(*iii*)

(Partial Fractions)

Multiplying both sides by L.C.M. = $(x + 1)(x + 3)$,

$$-20x - 15 = A(x + 3) + B(x + 1)$$
$$= Ax + 3A + Bx + B$$

Comparing coefficients of x and constant terms on both sides, we have

Coefficients of *x*: $A + B = -20$ ...(*iv*)

Constant terms: $3A + B = -15$ ...(*v*)

Subtracting (iv) and (v), $-2A = -5$. Therefore $A = \frac{5}{2}$.

Putting $A = \frac{5}{2}$ in (iv), $\frac{5}{2} + B = -20 \Rightarrow B = -20 - \frac{5}{2}$

or $B = \frac{-40-5}{2} = \frac{-45}{2}$

Putting these values of A and B in (iii),

$$\frac{-20x-15}{(x+1)(x+3)} = \frac{\frac{5}{2}}{x+1} - \frac{\frac{45}{2}}{x+3}$$

$$\therefore \quad I = \int_1^2 \frac{-20x-15}{(x+1)(x+3)}\,dx = \frac{5}{2}\int_1^2 \frac{1}{x+1}\,dx - \frac{45}{2}\int_1^2 \frac{1}{x+3}\,dx$$

$$= \frac{5}{2}\left(\log|x+1|\right)_1^2 - \frac{45}{2}\left(\log|x+3|\right)_1^2$$

$$= \frac{5}{2}\left(\log|3| - \log|2|\right) - \frac{45}{2}\left(\log|5| - \log|4|\right)$$

$$= \frac{5}{2}\log\frac{3}{2} - \frac{45}{2}\log\frac{5}{4} \qquad [\because |x| = x \text{ if } x \geq 0]$$

$$= \frac{5}{2}\left(\log\frac{3}{2} - 9\log\frac{5}{4}\right)$$

Putting this value of I in (ii),

$$\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx = 5 + \frac{5}{2}\left(\log\frac{3}{2} - 9\log\frac{5}{4}\right) = 5 - \frac{5}{2}\left(9\log\frac{5}{4} - \log\frac{3}{2}\right)$$

17. $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx$

Sol. $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx = 2\int_0^{\frac{\pi}{4}} \sec^2 x\,dx + \int_0^{\frac{\pi}{4}} x^3\,dx + 2\int_0^{\frac{\pi}{4}} 1\,dx$

$$= 2\left(\tan x\right)_0^{\frac{\pi}{4}} + \left(\frac{x^4}{4}\right)_0^{\frac{\pi}{4}} + 2\left(x\right)_0^{\frac{\pi}{4}}$$

$$= 2\left(\tan\frac{\pi}{4} - \tan 0\right) + \left(\frac{\left(\frac{\pi}{4}\right)^4}{4} - 0\right) + 2\left(\frac{\pi}{4} - 0\right)$$

$\left(\frac{\pi^4}{256}\right)$ ... $\frac{2\pi}{4}$

$= 2(1-0) + \frac{\left(\frac{\pi}{4}\right)^4}{4} + \frac{\pi}{4} \qquad = 2 + \frac{\pi^4}{1024} + \frac{\pi}{2}$.

18. $\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$

Sol. $\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx = \int_0^{\pi}\left[\left(\frac{1-\cos x}{2}\right) - \left(\frac{1+\cos x}{2}\right)\right]dx$

$\left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2} \text{ and } \cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$

$$= \int_0^{\pi} \left(\frac{1 - \cos x - 1 - \cos x}{2} \right) dx = \int_0^{\pi} \frac{-2\cos x}{2} dx$$

$$= -\int_0^{\pi} \cos x \, dx = -(\sin x)_0^{\pi} = -(\sin \pi - \sin 0) = -(0 - 0) = 0.$$

$[\because \sin \pi = \sin 180° = \sin (180° - 0) = \sin 0 = 0]$

19. $\int_0^2 \frac{6x + 3}{x^2 + 4} dx$

Sol. $\int_0^2 \frac{6x + 3}{x^2 + 4} dx = \int_0^2 \frac{6x}{x^2 + 4} dx + 3 \int_0^2 \frac{1}{x^2 + 4} dx$

$$= 3 \int_0^2 \frac{2x}{x^2 + 4} dx + 3 \frac{1}{2} \left(\tan^{-1} \frac{x}{2} \right)_0^2$$

$$= 3 \left(\log | x^2 + 4 | \right)_0^2 + \frac{3}{2} (\tan^{-1} 1 - \tan^{-1} 0)$$

$$\left[\because \int \frac{f'(x)}{f(x)} dx = \log | f(x) | \text{ and } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$$

$$= 3 (\log 8 - \log 4) + \frac{3}{2} \left(\frac{\pi}{4} - 0 \right) \quad \left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$= 3 \log \frac{8}{4} + \frac{3\pi}{8} = 3 \log 2 + \frac{3\pi}{8}.$$

20. $\int_0^1 \left(x e^x + \sin \frac{\pi x}{4} \right) dx$

Sol. $\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx = \int_0^1 \underset{\text{I}}{x} \, \underset{\text{II}}{e^x} dx + \int_0^1 \sin \frac{\pi x}{4} dx$

Applying Product Rule on first definite integral,

$$\left(\text{I} \int \text{II} \, dx \right)_0^1 - \int_0^1 \left(\frac{d}{dx} (\text{I}) \int \text{II} \, dx \right) dx$$

$$= \left(x e^{x}\right)_0^1 - \int_0^1 1.\, e^{x}\, dx - \frac{\left(\cos \frac{\pi x}{4}\right)_0^1}{\frac{\pi}{4}} \rightarrow \text{Coefficient of } x \text{ in } \frac{\pi x}{4}$$

$$= e^1 - 0 - \int_0^1 e^x\, dx - \frac{4}{\pi}\left[\cos\frac{\pi}{4} - \cos 0\right] = e - \left(e^x\right)_0^1 - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= e - (e - e^0) - \frac{4}{\pi\sqrt{2}} + \frac{4}{\pi}$$

$$= e - e + 1 - \frac{2.2}{\pi\sqrt{2}} + \frac{4}{\pi} = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}.$$

Choose the correct answer in Exercises 21 and 22:

21. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ **equals**

(A) $\frac{\pi}{3}$ **(B)** $\frac{2\pi}{3}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{12}$

Sol. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{4} \qquad \left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \tan\frac{\pi}{4} = 1\right]$

$= \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$

$\therefore$ Option (D) is the correct answer.

22. $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ **equals**

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{12}$ **(C)** $\frac{\pi}{24}$ **(D)** $\frac{\pi}{4}$

Sol. $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = \int_0^{\frac{2}{3}} \frac{dx}{(3x)^2+2^2} = \left[\frac{\frac{1}{2}\tan^{-1}\frac{3x}{2}}{3}\right]$ → Coefficient of x in $3x$

$= \frac{1}{6}\left[\tan^{-1}\frac{3x}{2}\right]_0^{\frac{2}{3}} = \frac{1}{6}\left[\tan^{-1}\left(\frac{3}{2}\times\frac{2}{3}\right) - \tan^{-1} 0\right]$

$= \frac{1}{6}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{1}{6}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{24}$

$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$

$\therefore$ Option (C) is the correct answer.

Exercise 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution:

1. $\int_0^1 \frac{x}{x^2+1}\,dx$

Sol. Let I $= \int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$ ...(*i*)

Put $x^2 + 1 = t$. Therefore $2x = \frac{dt}{dx} \Rightarrow 2x\,dx = dt$.

To change the limits of integration from values of *x* to values of *t*.

When $x = 0$, $t = 0 + 1 = 1$

When $x = 1$, $t = 1 + 1 = 2$

$\therefore$ From (*i*), I $= \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}(\log|t|)_1^2 = \frac{1}{2}(\log|2| - \log|1|)$

$$= \frac{1}{2}(\log 2 - \log 1) = \frac{1}{2}(\log 2 - 0) = \frac{1}{2}\log 2.$$

2. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\, d\phi$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\, d\phi$ ...(*i*)

Put $\sin\phi = t$.

($\because$ one factor of integrand is $\cos^5\phi$ where $n = 5$ is odd.)

$\therefore \cos\phi = \frac{dt}{d\phi}$ *i.e.,* $\cos\phi\, d\phi = dt.$

To change the limits of integration from ϕ to t

When $\phi = 0$, $t = \sin\phi = \sin 0 = 0$

When $\phi = \frac{\pi}{2}$, $t = \sin\phi = \sin\frac{\pi}{2} = 1$

Now Integrand $\sqrt{\sin\phi}\,\cos^5\phi = \sqrt{\sin\phi}\,\cos^4\phi\cos\phi$

$$= \sqrt{\sin\phi}\,(\cos^2\phi)^2\cos\phi = \sqrt{\sin\phi}\,(1-\sin^2\phi)^2\cos\phi$$

$\therefore$ From (*i*), $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,(1-\sin^2\phi)^2\cos\phi\, d\phi$

$$= \int_0^1 \sqrt{t}\,(1-t^2)^2\, dt = \int_0^1 t^{1/2}(1+t^4-2t^2)\, dt$$

$$= \int_0^1 \left(t^{\frac{1}{2}} + t^{\frac{1}{2}+4} - 2t^{\frac{1}{2}+2}\right) dt = \int_0^1 (t^{1/2} + t^{9/2} - 2t^{5/2})\, dt$$

$$= \int_0^1 t^{1/2}\, dt + \int_0^1 t^{9/2}\, dt - 2\int_0^1 t^{5/2}\, dt$$

$$= \frac{\left(t^{3/2}\right)_0^1}{\frac{3}{2}} + \frac{\left(t^{11/2}\right)_0^1}{\frac{11}{2}} - 2\frac{\left(t^{7/2}\right)_0^1}{\frac{7}{2}}$$

$$= \frac{2}{3}(1-0) + \frac{2}{11}(1-0) - \frac{4}{7}(1-0)$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} = \frac{2(77)+2(21)-4(33)}{3(11)(7)}$$

$$= \frac{154+42-132}{}$$

$$231 \qquad = \frac{196-132}{231} = \frac{64}{231}.$$

3. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Sol. Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$ ...(*i*)

Put $x = \tan\theta$. $\therefore \frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta \, d\theta$

To change the limits of integration

When $x = 0$, $\tan\theta = 0 = \tan 0 \Rightarrow \theta = 0$

When $x = 1$, $\tan\theta = 1 = \tan\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore$ From (i), $I = \int_0^{\frac{\pi}{4}} \left(\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\right)\sec^2\theta\, d\theta$

$= \int_0^{\frac{\pi}{4}} (\sin^{-1}(\sin 2\theta))\sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sec^2\theta}\, d\theta$

Applying Product Rule of Integration

$$\int_a^b \text{I}\,.\,\text{II}\, dx = \left(\text{I}\int \text{II}\right)_a^b - \int_a^b \left(\frac{d}{dx}(\text{I})\int \text{II}\, dx\right) dx$$

$= 2\left[(\theta\,.\tan\theta)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 1\,.\tan\theta\, d\theta\right]$

$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - 0 - \int_0^{\frac{\pi}{4}}\tan\theta\, d\theta\right] = 2\left[\frac{\pi}{4} - (\log\sec\theta)_0^{\frac{\pi}{4}}\right]$

$= 2\left[\frac{\pi}{4} - \left(\log\sec\frac{\pi}{4} - \log\sec 0\right)\right] = 2\left[\frac{\pi}{4} - (\log\sqrt{2} - \log 1)\right]$

$= \frac{\pi}{2} - 2\log 2^{1/2}$ ($\because \log 1 = 0$)

$= \frac{\pi}{2} - 2\,.\,\frac{1}{2}\log 2 = \frac{\pi}{2} - \log 2.$

4. $\int_0^2 x\sqrt{x+2}\, dx$

Sol. Let $I = \int_0^2 x\sqrt{x+2}\, dx$ ...(i)

Put $\sqrt{\text{Linear}} = t$, *i.e.*, $\sqrt{x+2} = t$. Therefore $x + 2 = t^2$.

$\therefore \frac{dx}{dt} = 2t \Rightarrow dx = 2t\, dt$

To change the limits of Integration

When $x = 0$, $t = \sqrt{x+2}$ ... $\sqrt{4}$ ... ∴ From (i), I = $\int^2$

When $x = 2$, $t = \sqrt{x+2}$

$= \sqrt{2}$

$= \sqrt{2+2} = 2.$

$(t^2 - 2)$ $\quad \frac{t}{dt} \cdot 2t$

$[\because x + 2 = t^2 \Rightarrow x = t^2 - 2]$

$$= 2 \int_{\sqrt{2}}^{2} t^2(t^2 - 2)\, dt = 2 \int_{\sqrt{2}}^{2} (t^4 - 2t^2)\, dt$$

$$= 2 \left[\left(\frac{t^5}{5} \right) - 2\left(\frac{t^3}{3} \right) \right]_{\sqrt{2}}^{2} = 2 \left[\frac{1}{5}\left(2^5 - (\sqrt{2})^5\right) - \frac{2}{3}\left(2^3 - (\sqrt{2})^3\right) \right]$$

$$= 2\left[\frac{1}{5}(32 - 4\sqrt{2}) - \frac{2}{3}(8 - 2\sqrt{2})\right] \quad \left[\because (\sqrt{2})^3 = \sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2} = 2\sqrt{2},\right.$$

$$\left.\text{and } (\sqrt{2})^5 = \sqrt{2}.\sqrt{2}.\sqrt{2}.\sqrt{2}.\sqrt{2} = 4\sqrt{2}\right]$$

$$= 2\left[\frac{32}{5} - \frac{4\sqrt{2}}{5} - \frac{16}{3} + \frac{4\sqrt{2}}{3}\right] = 2\left[\frac{96 - 12\sqrt{2} - 80 + 20\sqrt{2}}{15}\right]$$

$$= \frac{2}{15}(16 + 8\sqrt{2}) = \frac{16}{15}(2 + \sqrt{2}) = \frac{16}{15}(\sqrt{2}\cdot\sqrt{2} + \sqrt{2})$$

$$= \frac{16\sqrt{2}}{15}(\sqrt{2} + 1).$$

5. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_0^{\frac{\pi}{2}} \frac{-\sin x}{1+\cos^2 x}\,dx$ ...(*i*)

Put $\cos x = t$. Therefore $-\sin x = \dfrac{dt}{dx} \Rightarrow -\sin x\,dx = dt.$

To change the limits of Integration.

When $x = 0$, $t = \cos 0 = 1$, When $x = \frac{\pi}{2}$, $t = \cos\frac{\pi}{2} = 0$

$\therefore$ From (*i*), $I = -\displaystyle\int_1^0 \frac{dt}{1+t^2} = -\int_1^0 \frac{1}{1+t^2}\,dt$

$$= -\left(\tan^{-1} t\right)_1^0 = -(\tan^{-1} 0 - \tan^{-1} 1) = -\left(0 - \frac{\pi}{4}\right)$$

$$\left[\because \tan 0 = 0 \Rightarrow \tan^{-1} 0 = 0 \text{ and } \tan\frac{\pi}{4} = 1 \Rightarrow \tan^{-1} 1 = \frac{\pi}{4}\right] = \frac{\pi}{4}.$$

6. $\displaystyle\int_0^2 \frac{dx}{x + 4 - x^2}$

Sol. $\displaystyle\int_0^2 \frac{dx}{4 + x - x^2} = \int_0^2 \frac{dx}{-x^2 + x + 4} = \int_0^2 \frac{dx}{-(x^2 - x - 4)}$

(Making coeff. of x^2 numerically unity)

Completing squares by adding and subtracting

$$\left(\frac{1}{2}\text{coeff. of } x\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \therefore = \int_0^2 \frac{dx}{-\left[x^2 - x + \frac{1}{4} - \frac{1}{4} - 4\right]}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2 - \frac{17}{4}\right]} = \int_0^2 \frac{dx}{\frac{17}{4} - \left(x-\frac{1}{2}\right)^2} = \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\times\frac{\sqrt{17}}{2}}\left[\log\left|\frac{\frac{\sqrt{17}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}-\left(x-\frac{1}{2}\right)}\right|\right]_0^2 \quad \left(\because \int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right)$$

$$= \frac{1}{\sqrt{17}}\left[\log\left|\frac{\sqrt{17}+2x-1}{\sqrt{17}-2x+1}\right|\right]_0^2$$

$$= \frac{1}{\sqrt{17}}\left[\log\left|\frac{\sqrt{17}+3}{\sqrt{17}-3}\right|-\log\left|\frac{\sqrt{17}-1}{\sqrt{17}+1}\right|\right]$$

$$= \frac{1}{\sqrt{17}}\log\left|\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}\right| = \frac{1}{\sqrt{17}}\log\frac{20+4\sqrt{17}}{20-4\sqrt{17}}$$

$(\because (\sqrt{17}+3)(\sqrt{17}+1) = 17+\sqrt{17}+3\sqrt{17}+3 = 20+4\sqrt{17}$. Similarly $(\sqrt{17}-3)(\sqrt{17}-1) = 20-4\sqrt{17})$

$$= \frac{1}{\sqrt{17}}\log\frac{4(5+\sqrt{17})}{4(5-\sqrt{17})} = \frac{1}{\sqrt{17}}\log\frac{5+\sqrt{17}}{5-\sqrt{17}}$$

$$= \frac{1}{\sqrt{17}}\log\left|\frac{5+\sqrt{17}}{5-\sqrt{17}}\times\frac{5+\sqrt{17}}{5+\sqrt{17}}\right| = \frac{1}{\sqrt{17}}\log\frac{(5+\sqrt{17})^2}{25-17}$$

$$= \frac{1}{\sqrt{17}}\log\frac{42+10\sqrt{17}}{8} = \frac{1}{\sqrt{17}}\log\frac{21+5\sqrt{17}}{4}.$$

7. $\displaystyle\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

Sol. Let $I = \displaystyle\int_{-1}^{1}\frac{dx}{x^2+2x+5} = \int_{-1}^{1}\frac{dx}{x^2+2x+1+4}$ (To complete squares)

$$= \int_{-1}^{1}\frac{1}{(x+1)^2+2^2}\,dx \qquad ...(i)$$

Put $x + 1 = t$. $\quad \therefore \frac{dx}{dt} = 1 \quad \Rightarrow dx = dt$

To change the limits of integration

When $x = -1,\ t = -1 + 1 = 0$

When $x = 1,\ t = 1 + 1 = 2$

$\therefore$ From (*i*), $I = \int_0^2 \frac{1}{t^2 + 2^2}\, dt = \frac{1}{2}\left(\tan^{-1}\frac{t}{2}\right)_0^2$ $\left[\because \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$= \frac{1}{2}\left[\tan^{-1}\frac{2}{2} - \tan^{-1}\frac{0}{2}\right] = \frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1} 0\right)$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}. \qquad \left[\because \tan\frac{\pi}{4} = 1 \text{ and } \tan 0 = 0\right]$$

8. $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx$

Sol. Let $I = \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\, dx$ ...(*i*)

[Type $\int (f(x) + g(x))\ e^{ax}\, dx$. Put $ax = t$ and it will become

$\int (f(t) + f'(t))\ e^t\, dt = e^t f(t)$]

Put $2x = t$ $\therefore$ $2 = \frac{dt}{dx}$ $\Rightarrow$ $2dx = dt$ $\Rightarrow$ $dx = \frac{dt}{2}$

To change the limits of Integration

When $x = 1$, $t = 2x = 2$, When $x = 2$, $t = 2x = 4$

$\therefore$ From (*i*), $I = \int_2^4 \left(\frac{1}{\frac{t}{2}} - \frac{1}{2\left(\frac{t}{2}\right)^2}\right) e^t \frac{dt}{2}$ $\left[\because 2x = t \Rightarrow x = \frac{t}{2}\right]$

$\therefore$ $I = \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t \frac{dt}{2} = \int_2^4 \frac{1}{2}.2\left(\frac{1}{t} - \frac{1}{t^2}\right) e^t\, dt$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t\, dt = \int_2^4 (f(t) + f'(t))\ e^t\, dt$$

$\left(\text{Here } f(t) = \frac{1}{t} = t^{-1} \text{ and therefore } f'(t) = (-1)t^{-2} = \frac{-1}{t^2}\right)$

$$= \left(e^t f(t)\right)_2^4 = \left(\frac{e^t}{t}\right)_2^4 = \frac{e^4}{4} - \frac{e^2}{2} = \frac{e^4 - 2e^2}{4} = \frac{e^2(e^2 - 2)}{4}.$$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{1/3}}{x^4}\, dx$ is

(A) 6 **(B) 0** **(C) 3** **(D) 4**

Sol. Let $I = \int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{1/3}}{x^4}\, dx$

$\left[x^3\left(\frac{x}{} - 1\right)\right]^{1/3}$

$$= \int_{\frac{1}{3}}^{1} \frac{(x^3)^{1/3}\left(\frac{1}{x^3} - 1\right)^{1/3}}{x^4}\, dx = \int_{\frac{1}{3}}^{1} \frac{x\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^4}\, dx$$

$$= \int_{\frac{1}{3}}^{1} \frac{x\,(x^{-2} - 1)^{1/3}}{x^4}\, dx = \int_{\frac{1}{3}}^{1} \frac{(x^{-2} - 1)^{1/3}}{x^3}\, dx$$

$$I = \frac{-1}{2} \int_{\frac{1}{3}}^{1} (x^{-2} - 1)^{1/3} (-2x^{-3})\, dx \quad \ldots(i)$$

Put $x^{-2} - 1 = t$

Therefore $-2x^{-3} = \frac{dt}{dx} \Rightarrow -2x^{-3}\,dx = dt$

To change the limits of Integration

When $x = \frac{1}{3}$, $t = x^{-2} - 1 = \left(\frac{1}{3}\right)^{-2} - 1$

$= (3^{-1})^{-2} - 1 = 3^2 - 1 = 9 - 1 = 8$

When $x = 1$, $t = 1^{-2} - 1 = 1 - 1 = 0$

$\therefore$ From (i), $I = \frac{-1}{2}\int_8^0 t^{1/3}\,dt = \frac{-1}{2}\left(\frac{t^{4/3}}{\frac{4}{3}}\right)_8^0$

$= \frac{-1}{2}\cdot\frac{3}{4}[0 - 8^{4/3}] = \frac{-3}{8}[-(2^3)^{4/3}] = \frac{-3}{8}(-2^4) = \frac{3}{8} \times 16 = 6$

$\therefore$ Option (A) is the correct answer.

10. If $f(x) = \int_0^x t\sin t\;dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$ **(B) $x\sin x$**

(C) $x\cos x$ **(D) $\sin x + x\cos x$**

Sol. $f(x) = \int_0^x \underset{\text{I}}{t}\,\underset{\text{II}}{\sin t}\;dt$

Applying Product Rule of Integration

$$\left[\int_a^b \text{I}\,.\,\text{II}\;dx = \left(\text{I}\int \text{II}\;dx\right)_a^b - \int_a^b \frac{d}{dx}(\text{I})\int \text{II}\;dx\,dx\right]$$

$\Rightarrow f(x) = \left(t(-\cos t)\right)_0^x - \int_0^x 1(-\cos t)\,dt$

$= -x\cos x - 0 + \int_0^x \cos t\;dt = -x\cos x + (\sin t)_0^x$

$= -x\cos x + \sin x - \sin 0 = -x\cos x + \sin x$

$\therefore f'(x) = -(x(-\sin x) + (\cos x)1) + \cos x$

$= x\sin x - \cos x + \cos x = x\sin x$

$\therefore$ Option (B) is the correct answer.

OR

$f(x) = \int_0^x \sin t\,dt$ $\therefore f'(x) = (t\sin t)_0^x$

[$\because$ Derivative operator and integral operator cancel with each other]

$= x\sin x - 0 = x\sin x$

Exercise 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 6:

1. $\int_0^{\frac{\pi}{2}} \cos^2 x \; dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$ ...(*i*)

$\therefore \quad I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $\quad I = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*),

$$2I = \int_0^{\frac{\pi}{2}} (\cos^2 x + \sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} 1 \, dx = (x)_0^{\frac{\pi}{2}}$$

$\Rightarrow \quad 2I = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}.$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, dx$ ...(*i*)

$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} \, dx$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $\quad I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \, dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right) dx = \int_0^{\frac{\pi}{2}} 1 \, dx$$

$\Rightarrow 2I = (x)_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \Rightarrow \quad I = \frac{\pi}{4}.$

4

3. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x\, dx}{\sin^{3/2} x + \cos^{3/2} x}$

Sol. Let $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\, dx$ ...(*i*)

Changing x to $\frac{\pi}{2} - x$ $\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2}\left(\frac{\pi}{2}-x\right)}{\sin^{3/2}\left(\frac{\pi}{2}-x\right)+\cos^{3/2}\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^{3/2} x}{\cos^{3/2} x+\sin^{3/2} x} \quad \text{...}(ii)$$

Adding Eqns. (i) and (ii),

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x+\cos^{3/2} x}{\sin^{3/2} x+\cos^{3/2} x}\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4}.$$

4. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^5 x\,dx}{\sin^5 x+\cos^5 x}$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x+\cos^5 x}\,dx$ ...(i)

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

or $\quad I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x+\sin^5 x}\,dx$ ...(ii)

Adding Eqns. (i) and (ii), we have

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\cos^5 x}{\sin^5 x+\cos^5 x}+\frac{\sin^5 x}{\cos^5 x+\sin^5 x}\right)dx$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x+\sin^5 x}{\sin^5 x+\cos^5 x}\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = (x)_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \qquad \Rightarrow \quad I = \frac{\pi}{4}.$$

5. $\displaystyle\int_{-5}^{5} |x+2|\,dx$

Sol. Let $I = \displaystyle\int_{-5}^{5} |x+2|\,dx$ ...(i)

We can evaluate this integral only if we can get rid of the modulus.

Putting expression within modulus equal to 0, we have

$x + 2 = 0$, *i.e.*, $x = -2 \in (-5, 5)$

$\therefore$ From (*i*), $I = \int_{-5}^{5} |x+2| dx$

$$= \int_{-5}^{-2} |x+2|\, dx + \int_{-2}^{5} |x+2|\, dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ where } a < c < b\right]$$

$$= \int_{-5}^{-2} -(x+2)\ dx + \int_{-2}^{5} (x+2)\ dx$$

[$\because$ On $(-5, -2)$, $x < -2 \Rightarrow x + 2 < 0$
$\Rightarrow |x + 2| = -(x + 2)$ and on $(-2, 5)$; $x > -2$
$\Rightarrow x + 2 > 0 \Rightarrow |x + 2| = x + 2$, by definition of modulus function]

$$= -\left(\frac{x^2}{2} + 2x\right)_{-5}^{-2} + \left(\frac{x^2}{2} + 2x\right)_{-2}^{5}$$

$$= -\left[\left(\frac{4}{2} - 4\right) - \left(\frac{25}{2} - 10\right)\right] + \left[\left(\frac{25}{2} + 10\right) - \left(\frac{4}{2} - 4\right)\right]$$

$$= -\left[-2 - \frac{5}{2}\right] + \left[\frac{45}{2} + 2\right] = 2 + \frac{5}{2} + \frac{45}{2} + 2$$

$$= 4 + \frac{50}{2} = 4 + 25 = 29.$$

6. $\int_2^8 |x - 5|\ dx$

Sol. We know by definition of modulus function, that

$$|x - 5| = \begin{cases} x - 5 & \text{if } x - 5 \geq 0, \text{ i.e., } x \geq 5 \quad ...(i) \\ -(x-5) = 5 - x, & \text{if } x < 5 \quad ...(ii) \end{cases}$$

$$\therefore \int_2^8 |x-5|\ dx = \int_2^5 |x-5|\ dx + \int_5^8 |x-5|\ dx$$

$$= \int_2^5 (5 - x)\ dx + \int_5^8 (x - 5)\ dx = \left(5x - \frac{x^2}{2}\right)_2^5 + \left(\frac{x^2}{2} - 5x\right)_5^8$$

[By (*ii*)] [By (*i*)]

$$= \left(25 - \frac{25}{2}\right) - (10 - 2) + (32 - 40) - \left(\frac{25}{2} - 25\right)$$

$$= 25 - \frac{25}{2} - 8 - 8 - \frac{25}{2} + 25 \qquad = 34 - \frac{50}{2} = 34 - 25 = 9$$

By using the properties of definite integrals, evaluate the

integrals in Exercises 7 to 11:

7. $\int_0^1 x(1-x)^n \, dx$

Sol. Let $I = \int_0^1 x(1-x)^n \, dx$

$$\therefore \quad I = \int_0^1 (1-x)\,(1-(1-x))^n \, dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

or $\quad I = \int_0^1 (1-x)\,(1-1+x)^n \, dx$

or $\quad I = \int_0^1 (1-x)\, x^n\, dx = \int_0^1 (x^n - x^{n+1})\, dx$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \frac{1}{n+1} - \frac{1}{n+2} - (0-0)$$

$$= \frac{n+2-n-1}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}.$$

8. $\displaystyle\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx \qquad ...(i)$

Changing x to $\frac{\pi}{4} - x$ $\qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$I = \int_0^{\frac{\pi}{4}} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right] dx = \int_0^{\frac{\pi}{4}} \log\left[1 + \frac{1-\tan x}{1+\tan x}\right] dx$$

$$\left[\because \tan\left(\frac{\pi}{4} - x\right) = \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x} = \frac{1-\tan x}{1+\tan x}\right]$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan x + 1 - \tan x}{1+\tan x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right) dx \qquad ...(ii)$$

Adding Eqns. (i) and (ii), we have

$$2I = \int_0^{\frac{\pi}{4}} \left[\log(1+\tan x) + \log\left(\frac{2}{1+\tan x}\right)\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[(1+\tan x)\frac{2}{(1+\tan x)}\right] dx = \int_0^{\frac{\pi}{4}} \log 2\, dx$$

or $\quad 2I = (\log 2)\,[x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log 2$. Dividing by 2, $I = \frac{\pi}{8}\log 2$.

9. $\int_0^2 x\sqrt{2-x}\ dx$

Sol. Let $I = \int_0^2 x\sqrt{2-x}\ dx$

Changing x to $2 - x$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$I = \int_0^2 (2-x)\sqrt{2-(2-x)}\ dx$$

$$= \int_0^2 (2-x)\sqrt{x}\ dx = \int_0^2 (2x^{1/2} - x^{3/2})\ dx$$

$$= \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\right]_0^2 = \left(\frac{4}{3} \cdot 2^{3/2} - \frac{2}{5} \cdot 2^{5/2}\right) - (0 - 0)$$

$$= \frac{4}{3} \times 2\sqrt{2} - \frac{2}{5} \times 4\sqrt{2} = \left(\frac{8}{3} - \frac{8}{5}\right)\sqrt{2}$$

$$\left(\because\ 2^{3/2} = (2^{1/2})^3 = (\sqrt{2})^3 = \sqrt{2}\,\sqrt{2}\,\sqrt{2} = 2\sqrt{2}\right.$$

and $2^{5/2} = (2^{1/2})^5 = (\sqrt{2})^5 = \sqrt{2}\,\sqrt{2}\,\sqrt{2}\,\sqrt{2}\,\sqrt{2} = 2.2.\sqrt{2}$

$$\left. = 4\sqrt{2}\right) = \frac{16\sqrt{2}}{15}.$$

10. $\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x)\, dx$

Sol. Let $I = \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\, dx$

$$= \int_0^{\pi/2} (\log \sin^2 x - \log \sin 2x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{\sin 2x}\right) dx = \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{2 \sin x \cos x}\right) dx$$

or $I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan x\right) dx$ ...(*i*)

$\therefore$ $I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan\left(\frac{\pi}{2} - x\right)\right) dx$ $\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$

or $I = \int_0^{\pi/2} \log\left(\frac{1}{2} \cot x\right) dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*),

$$2I = \int_0^{\pi/2} \left[\log\left(\frac{1}{2} \tan x\right) + \log\left(\frac{1}{2} \cot x\right)\right] dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log\left(\frac{1}{2} \tan x \cdot \frac{1}{2} \cot x\right) dx = \int_0^{\pi/2} \log \frac{1}{4}\, dx = \log \frac{1}{4}\, (x)_0^{\pi/2}$$

$$= (\log 1 - \log 4)\frac{\pi}{2} = -\frac{\pi}{2} \log 4 \qquad (\because\ \log 1 = 0)$$

$\therefore \quad I = -\frac{\pi}{4} \log 4 = -\frac{\pi}{4} \log 2^2 = -\frac{2\pi}{4} \log 2 = -\frac{\pi}{2} \log 2.$

11. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$

Sol. Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$ or $I = 2\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ ...(*i*)

[$\because$ For $f(x) = \sin^2 x$, $f(-x) = \sin^2(-x) = (-\sin x)^2 = \sin^2 x = f(x)$

$\therefore$ $f(x)$ is an even function of x and hence

$$\int_{-a}^{a} f(x)\,dx = 2\int^{a} f(x)\,dx]$$

$$\therefore \quad I = 2\int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right) dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\text{or} \quad I = 2\int_0^{\frac{\pi}{2}} \cos^2 x\,dx \qquad ...(ii)$$

Adding Eqns. (*i*) and (*ii*), we have

$$2I = 2\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)\,dx$$

$$\text{or} \quad 2I = 2\int_0^{\frac{\pi}{2}} 1\,dx = 2\,(x)_0^{\frac{\pi}{2}} = 2\,.\,\frac{\pi}{2} = \pi \quad \therefore \quad I = \frac{\pi}{2}.$$

Using properties of definite integrals, evaluate the following integrals in Exercises 12 to 18:

12. $\displaystyle\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

Sol. Let $I = \displaystyle\int_0^{\pi} \frac{x}{1+\sin x}\,dx$ ...(*i*)

Changing x to $\pi - x$, $I = \displaystyle\int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx$

$$\text{or} \quad I = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx \quad ...(ii) \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^{\pi}\left(\frac{x}{1+\sin x} + \frac{\pi - x}{1+\sin x}\right) dx = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x}\,dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin x}\,dx = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$

$$\text{or} \quad 2I = 2\pi\int_0^{\pi/2} \frac{dx}{1+\sin x}$$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$$

$$= 2\pi\int_0^{\pi/2} \frac{dx}{1+\sin\left(\frac{\pi}{2} - x\right)} \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$= 2\pi$

$$\int_0^{\pi/2} \frac{dx}{1+\cos x} \qquad \int \left[\right]$$

$$\left[\tan \frac{x}{2}\right]_0^{\pi/2}$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{x}{2}} = \frac{\pi}{2} \int_0^{\pi/2} \sec^2 \frac{x}{2}\, dx = \frac{\pi}{2} \left[\frac{2}{1} \right]_0$$

$$= \pi\left(\tan\frac{\pi}{4} - \tan 0\right) = \pi(1-0) = \pi.$$

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \; dx$

Sol. Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \; dx$

Here Integrand $f(x) = \sin^7 x$

$\therefore\ f(-x) = \sin^7(-x) = (-\sin x)^7 = -\sin^7 x = -f(x)$

$\therefore\ f(x)$ is an odd function of x.

$$\therefore \quad I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \; dx = 0.$$

$$\left[\because \text{ If } f(x) \text{ is an odd function of } x, \text{ then } \int_{-a}^{a} f(x)\,dx = 0\right]$$

14. $\int_0^{2\pi} \cos^5 x \; dx$

Sol. $\int_0^{2\pi} \cos^5 x \; dx = 2\int_0^{\pi} \cos^5 x \; dx$

$$\left[\because \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$$

Here $f(x) = \cos^5 x \quad \therefore \quad f(2\pi - x) = \cos^5(2\pi - x) = \cos^5 x$

$= f(x) = 2(0) = 0$

$\left[\because \int_0^{2a} f(x)\; dx = 0, \text{ if } f(2a-x) = -f(x). \text{ Here } f(x) = \cos^5 x\right.$

$\left.\therefore\ f(\pi - x) = \cos^5(\pi - x) = (-\cos x)^5 = -\cos^5 x = -f(x)\right]$

Alternatively. To evaluate $\int_0^{2\pi} \cos^5 x \; dx$, put $\sin x = t$.

Remark. In fact $\int_0^{2\pi} \cos^n x \; dx$ or $\int_0^{\pi} \cos^n x \; dx$ for all positive **odd integers** n is equal to zero.

This is a very important result for I.I.T. Entrance Examination.

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ ...(*i*)

Changing x to $\frac{\pi}{2} - x$ in integrand of (*i*),

$$\left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right]$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx \qquad ...(ii)$$

Adding equations (*i*) and (*ii*), we have 2I = 0 or I = 0.

16. $\int_0^{\pi} \log(1 + \cos x)\, dx$

Sol. Let $I = \int_0^{\pi} \log(1 + \cos x)\, dx \qquad ...(i)$

$\therefore \quad I = \int_0^{\pi} \log(1 + \cos(\pi - x))\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$

or $\quad I = \int_0^{\pi} \log(1 - \cos x)\, dx \qquad ...(ii)$

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^{\pi} [\log(1 + \cos x) + \log(1 - \cos x)]\, dx$$

$$= \int_0^{\pi} \log((1 + \cos x)(1 - \cos x))\, dx \quad = \int_0^{\pi} \log(1 - \cos^2 x)\, dx$$

$$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x\, dx = 2\int_0^{\pi} \log \sin x\, dx \quad (\because \log m^n = n \log m)$$

Dividing by 2, $I = \int_0^{\pi} \log \sin x\, dx = 2\int_0^{\frac{\pi}{2}} \log \sin x\, dx \qquad ...(iii)$

$\left[\because \text{For } f(x) = \log \sin x,\ f(\pi - x) = \log \sin(\pi - x) = \log \sin x = f(x) \text{ and if } f(2a - x) = f(x); \text{ then } \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx\right]$

$\therefore \quad I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$

or $\quad I = 2\int_0^{\frac{\pi}{2}} \log \cos x\, dx \qquad ...(iv)$

Adding Eqns. (*iii*) and (*iv*), we have

$$2I = 2\int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x)\, dx$$

Dividing by 2, $I = \int_0^{\frac{\pi}{2}} (\log \sin x \cos x)\, dx$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{2\sin x\cos x}{2}\right) dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx$$

or $$I = \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2)\, dx$$

or $$I = \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \int_0^{\frac{\pi}{2}} \log 2\, dx$$

or $$I = \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \log 2\, (x)_0^{\frac{\pi}{2}}$$

or $\quad I = \int_0^{\frac{\pi}{2}} \log \sin 2x \; dx - \frac{\pi}{2} \log 2$

or $\quad I = I_1 - \frac{\pi}{2} \log 2$ ...(*v*)

where $\quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin 2x \; dx$ ...(*vi*)

Put $2x = t$ to make I_1 look as I given by (*iii*)

$\therefore \quad 2 = \frac{dt}{dx}$ or $\quad 2\,dx = dt \quad$ or $\quad dx = \frac{dt}{2}$

To change the limits: When $x = 0, \quad t = 2x = 0$

When $x = \frac{\pi}{2}, \; t = 2x = \pi$

$\therefore \quad$ From (*vi*), $I_1 = \int_0^{\pi} \log \sin t \; \frac{dt}{2} = \frac{1}{2} \int_0^{\pi} \log \sin t \; dt$

or $\quad I_1 = \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log \sin t \, dt$

(For reason see Explanation within brackets below Eqn. (*iii*))

or $\quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin t \; dt = \int_0^{\frac{\pi}{2}} \log \sin x \, dx \left[\because \int_a^b f(t)\,dt = \int_a^b f(x)\,dx \right]$

or $\quad I_1 = \frac{I}{2}$ [By Eqn. (*iii*)]

Putting this value of I_1 in Eqn. (*v*), $I = \frac{I}{2} - \frac{\pi}{2} \log 2$

Multiplying by L.C.M. = 2, $2I = I - \pi \log 2$

or $2I - I = -\pi \log 2$ or $I = -\pi \log 2$.

17. $\displaystyle \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} \, dx$

Sol. Let $\quad I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} \, dx$ ...(*i*)

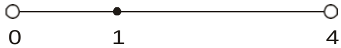
$\therefore \quad I = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{a - (a - x)}} \, dx = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} \, dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we have

$$2I = \int_0^a \left(\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} + \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}} \right) dx = \int_0^a \left(\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}} \right) dx$$

$$\text{or } 2I = \int_0^a 1 \, dx = (x)_0^a = a \quad \therefore \quad I = \frac{a}{2}.$$

18. $\int_0^4 |x-1|\, dx$



Sol. Let $I = \int_0^4 |x-1|\, dx$ ...(*i*)

Putting the expression $(x-1)$ within modulus equal to zero, we have

$$x = 1 \in (0, 4)$$

$\therefore$ From (*i*), $I = \int_0^4 |x-1|\, dx = \int_0^1 |x-1|\, dx + \int_1^4 |x-1|\, dx$

$$= -\int_0^1 (x-1)\, dx + \int_1^4 (x-1)\, dx$$

[$\because$ On (0, 1); $x < 1 \Rightarrow x - 1 < 0$ and hence $|x-1| = -(x-1)$ and on (1, 4), $x > 1 \Rightarrow x - 1 > 0$ and hence $|x-1| = (x-1)$ by definition of modulus function]

$$= -\left(\frac{x^2}{2} - x\right)_0^1 + \left(\frac{x^2}{2} - x\right)_1^4 = -\left(\left(\frac{1}{2} - 1\right) - 0\right) + \left(\frac{16}{2} - 4 - \left(\frac{1}{2} - 1\right)\right)$$

$$= \frac{-1}{2} + 1 + 8 - 4 - \frac{1}{2} + 1 \qquad = 6 - \frac{2}{2} = 6 - 1 = 5.$$

19. Show that $\int_0^a f(x)\, g(x)\, dx = 2\int_0^a f(x)\, dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$.

Sol. Given: $f(x) = f(a-x)$ ...(*i*)

and $g(x) + g(a-x) = 4$ ...(*ii*)

Let $I = \int_0^a f(x)\, g(x)\, dx$ ...(*iii*)

$\therefore$ $I = \int_0^a f(a-x)\, g(a-x)\, dx$ $\left[\because \int_0^a F(x)\, dx = \int_0^a F(a-x)\, dx\right]$

Putting $f(a-x) = f(x)$ from (*i*),

$$I = \int_0^a f(x)\, g(a-x)\, dx \qquad \text{...(iv)}$$

Adding Eqns. (*iii*) and (*iv*), we have

$2I = \int_0^a (f(x)\, g(x) + f(x)\, g(a-x))\, dx = \int_0^a f(x)\,(g(x) + g(a-x))\, dx$

or $2I = \int_0^a f(x)\,(4)\, dx$ [By (*ii*)] $= 4\int_0^a f(x)\, dx$

Dividing by 2, I $= 2\int_0^a f(x)\,dx$ = R.H.S.

Choose the correct answer in Exercises 20 and 21:

20. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\,dx$ is

(A) 0 **(B)** 2 **(C)** π **(D)** 1

Sol. Let I $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\,dx$

0

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, dx$$

$$= 0 + 0 + 0 + \left(x\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \left(\frac{-\pi}{2}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$\left[\because\right.$ Each of the three functions x^3, $x\cos x$ and $\tan^5 x$ is an odd function of x as $f(-x) = -f(x)$ for each of them and $\int_{-a}^{a} f(x)\,dx = 0$ for each odd function $f(x)\left.\right]$

$\therefore$ Option (C) is the correct option.

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(A) 2 **(B) $\frac{3}{4}$** **(C) 0** **(D) – 2**

Sol. Let $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ ...(*i*)

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right) dx$$

or $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$ ...(*ii*)

Adding Eqns. (*i*) and (*ii*), we get

$$2I = \int_0^{\frac{\pi}{2}} \left[\log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \log\left(\frac{4+3\cos x}{4+3\sin x}\right)\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left[\frac{4+3\sin x}{4+3\cos x}\cdot\frac{4+3\cos x}{4+3\sin x}\right] dx = \int_0^{\frac{\pi}{2}} \log 1 \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 \, dx = 0 \quad \Rightarrow \quad I = \frac{0}{2} = 0.$$

MISCELLANEOUS EXERCISE

Integrate the functions in Exercises 1 to 11:

1. $\dfrac{1}{x - x^3}$

Sol. The integrand $\dfrac{1}{x - x^3}$ is a rational function of x and the denominator $x - x^3 = x(1 - x^2) = x(1 - x)(1 + x)$ is the product of more than one factor. So, will form partial fractions.

$$\frac{1}{x - x^3} = \frac{1}{x(1 - x^2)} = \frac{1}{x(1 - x)(1 + x)}$$

$$= \frac{A}{x} + \frac{B}{1 - x} + \frac{C}{1 + x} \quad \ldots(i)$$

Multiplying every term of Eqn. (i) by L.C.M. $= x(1 - x)(1 + x)$,

$1 = A(1 - x)(1 + x) + Bx(1 + x) + Cx(1 - x)$

or $1 = A(1 - x^2) + B(x + x^2) + C(x - x^2)$

$\Rightarrow 1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$

Comparing coefficients of like powers on both sides,

x^2: $-A + B - C = 0$ ...(ii)

x: $B + C = 0$ .... (iii)

Constants: $A = 1$

Putting $A = 1$ in (ii), $-1 + B - C = 0$ or $B - C = 1$ ....(iv)

Adding Eqns. (iii) and (iv), $2B = 1 \Rightarrow B = \frac{1}{2}$

From (iii), $C = -B = \frac{-1}{2}$

Putting these values of A, B, C in (i),

$$\frac{1}{x - x^3} = \frac{1}{x} + \frac{\frac{1}{2}}{1 - x} - \frac{\frac{1}{2}}{1 + x}$$

$$\therefore \int \frac{1}{x - x^3}\, dx = \int \frac{1}{x}\, dx + \frac{1}{2}\int \frac{1}{1 - x}\, dx - \frac{1}{2}\int \frac{1}{1 + x}\, dx$$

$$= \log|x| + \frac{1}{2}\frac{\log|1 - x|}{-1} - \frac{1}{2}\log|1 + x|$$

$$= \frac{1}{2}[2\log|x| - \log|1 - x| - \log|1 + x|] + C$$

$$= \frac{1}{2}[\log|x|^2 - (\log|1 - x| + \log|1 + x|] + C$$

$$= \frac{1}{2}[\log|x|^2 - \log|1 - x||1 + x|] + C$$

$$= \frac{1}{2}[\log|x|^2 - \log|1 - x^2|] + C \qquad = \frac{1}{2}\log\left|\frac{x^2}{1 - x^2}\right| + C.$$

$\frac{1}{}$

2. $\sqrt{x+a}+\sqrt{x+b}$

Sol. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,dx$

Rationalising, $= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a}+\sqrt{x+b})(\sqrt{x+a}-\sqrt{x+b})}\,dx$

$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{x+a-(x+b)}\,dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}\,dx$

$[\because x+a-(x+b) = x+a-x-b = a-b]$

$= \frac{1}{a-b}\int (\sqrt{x+a}-\sqrt{x+b})\,dx$

$= \frac{1}{a-b}\left[\int (x+a)^{1/2}\,dx - \int (x+b)^{1/2}\,dx\right]$

$= \frac{1}{a-b}\left[\frac{(x+a)^{3/2}}{\frac{3}{2}(1)} - \frac{(x+b)^{3/2}}{\frac{3}{2}(1)}\right] + C$

$= \frac{1}{a-b}\left[\frac{2}{3}(x+a)^{3/2} - \frac{2}{3}(x+b)^{3/2}\right] + C$

$= \frac{2}{3(a-b)}[(x+a)^{3/2} - (x+b)^{3/2}] + C.$

3. $\frac{1}{x\sqrt{ax-x^2}}$

Sol. $I = \int \frac{dx}{x\sqrt{ax-x^2}}$ $\left[\textbf{Form } \int \frac{dx}{\text{Linear}\sqrt{\text{Quadratic}}}\right]$

Put Linear $= \frac{1}{t}$, *i.e.*, $x = \frac{1}{t} = t^{-1}$.

Differentiating both sides $dx = -\frac{1}{t^2}\,dt$

$\therefore \quad I = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{a}{t}-\frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{at-1}}$

$= -\int (at-1)^{-1/2}\,dt = -\frac{(at-1)^{1/2}}{\frac{1}{2}\times a} + c$

$= -\frac{2}{a}\sqrt{\frac{a}{x}-1} + c = -\frac{2}{a}\sqrt{\frac{a-x}{x}} + c.$

4. $\frac{1}{x^2(x^4+1)^{3/4}}$

Sol. $I = \int \frac{dx}{(x^4+1)^{3/4}} = \int \frac{dx}{\left[x^4\left(1+\frac{1}{x^4}\right)\right]^{3/4}} = \int \frac{dx}{x^3\left(1+\frac{1}{x^4}\right)^{3/4}}$

$\left[\because (x^4)^{\frac{3}{4}} = x^3\right]$

$$= \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-3/4} dx$$

Put $1 + \frac{1}{x^4} = t$ or $1 + x^{-4} = t$.

Differentiating both sides, $-4x^{-5}\, dx = dt$

or $-\frac{4}{x^5}\, dx = dt$ or $\frac{1}{x^5}\, dx = -\frac{1}{4}\, dt$

$$\therefore \quad I = -\frac{1}{4}\int t^{-3/4}\, dt = -\frac{1}{4} \cdot \frac{t^{1/4}}{1/4} + c \quad = -\left(1+\frac{1}{x^4}\right)^{1/4} + c.$$

5. $\dfrac{1}{x^{1/2} + x^{1/3}}$

Sol. Here the denominators of fractional powers $\frac{1}{2}$ and $\frac{1}{3}$ of x are 2 and 3. L.C.M. of 2 and 3 is 6.

Put $x = t^6$. Differentiating both sides, $dx = 6t^5\, dt$

$$\therefore \quad I = \int \frac{dx}{x^{1/2} + x^{1/3}} = \int \frac{6t^5}{t^3 + t^2}\, dt = 6\int \frac{t^5}{t^2\,(t+1)}\, dt$$

$$= 6\int \frac{t^3}{t+1}\, dt = 6\int \frac{(t^3+1)-1}{t+1}\, dt = 6\int \left[\frac{t^3+1}{t+1} - \frac{1}{t+1}\right] dt$$

$$= 6\int \left[\frac{(t+1)(t^2-t+1)}{t+1} - \frac{1}{t+1}\right] dt = 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right) dt$$

$[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)]$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right] + c$$

$$= 2t^3 - 3t^2 + 6t - 6\log|t+1| + c$$

Putting $t = x^{1/6}$ $\quad (\because\ x = t^6 \Rightarrow t = x^{1/6})$

$$= 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log|x^{1/6} + 1| + c.$$

6. $\dfrac{5x}{(x+1)(x^2+9)}$

Sol. Let $I = \int \frac{5x}{(x+1)(x^2+9)}\, dx$ ...(*i*)

Let $\frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$ ...(*ii*)

L.C.M. = $(x + 1)(x^2 + 9)$

Multiplying every term of (*ii*) by L.C.M.,

$5x = A(x^2 + 9) + (Bx + C)(x + 1)$

or $5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Comparing coefficients of x^2, x and constant terms on both sides,

x^2: $A + B = 0$ ...(*iii*)

x: $B + C = 5$ ...(*iv*)

Constant terms : $9A + C = 0$ ...(*v*)

Let us solve Eqns. (*iii*), (*iv*) and (*v*) for A, B, C.

(iii) – (*iv*) gives, (to eliminate B), $A - C = -5$ ...(*vi*)

Adding (*v*) and (*vi*), $10A = -5$

$\therefore \quad A = \frac{-5}{10} = \frac{-1}{2}$

Putting $A = \frac{-1}{2}$ in (*iii*), $\frac{-1}{2} + B = 0 \quad \Rightarrow \quad B = \frac{1}{2}$

Putting $B = \frac{1}{2}$ in (*iv*), $\frac{1}{2} + C = 5 \quad \Rightarrow \quad C = 5 - \frac{1}{2} = \frac{9}{2}$

Putting these values of A, B, C in (*ii*),

$$\frac{5x}{(x+1)(x^2+9)} = \frac{\frac{-1}{2}}{x+1} + \frac{\frac{1}{2}x + \frac{9}{2}}{x^2+9}$$

$$\therefore \int \frac{5x}{(x+1)(x^2+9)}\,dx$$

$$= \frac{-1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{x}{x^2+9}\,dx + \frac{9}{2}\int \frac{1}{x^2+3^2}\,dx$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9}\,dx + \frac{9}{2}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3} + c$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}\tan^{-1}\frac{x}{3} + c$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + c.$$

($\because \; x^2 + 9 \geq 9 > 0$ and hence $|x^2+9| = x^2+9$)

7. $\dfrac{\sin x}{\sin (x - a)}$

Sol. $\displaystyle\int \frac{\sin x}{\sin(x-a)}\,dx = \int \frac{\sin(x-a+a)}{\sin(x-a)}\,dx$

$$= \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}\,dx$$

[$\because$ sin (A + B) = sin A cos B + cos A sin B]

$$= \int\left[\frac{\sin(x-a)\cos a}{\ldots} + \frac{\cos(x-a)\sin a}{\ldots}\right]dx \qquad \left[\because \frac{a+b}{\ldots} = \frac{a}{\ldots} + \frac{b}{\ldots}\right]$$

$$\int \left| \quad \sin(x-a) \qquad \sin(x-a) \quad \right| \qquad \left\lfloor \quad c \quad c \quad c \right\rfloor$$

$$= \int [\cos a + \sin a \cot(x-a)]\, dx = \int \cos a\, dx + \int \sin a \cot(x-a)\, dx$$

$$= \cos a \int 1\, dx + \sin a \int \cot(x-a)\, dx$$

$$= (\cos a)x + \sin a \frac{\log|\sin(x-a)|}{1} + c \left[\because \int \cot x\, dx = \log|\sin x|\right]$$

$$= x \cos a + \sin a \log|\sin(x-a)| + c.$$

8. $\dfrac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$

Sol. $\displaystyle\int \frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}\,dx = \int \frac{e^{\log x^5}-e^{\log x^4}}{e^{\log x^3}-e^{\log x^2}}\,dx$ $[\because\ n\log m=\log m^n]$

$\displaystyle = \int \frac{x^5-x^4}{x^3-x^2}\,dx$ $[\because\ e^{\log f(x)}=f(x)]$

$\displaystyle = \int \frac{x^4(x-1)}{x^2(x-1)}\,dx = \int x^2\,dx = \frac{x^3}{3}+c.$

9. $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

Sol. Let $\displaystyle I = \int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$ ...(*i*)

Put $\sin x = t$. Therefore $\cos x = \dfrac{dt}{dx}$ $\Rightarrow$ $\cos x\,dx = dt$

$\therefore$ From (*i*), $\displaystyle I = \int \frac{dt}{\sqrt{4-t^2}} = \int \frac{dt}{\sqrt{2^2-t^2}}$

$= \sin^{-1}\left(\dfrac{t}{2}\right)+c$ $\left[\because \displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}\right]$

$= \sin^{-1}\left[\dfrac{1}{2}\sin x\right]+c.$

10. $\dfrac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

Sol. Let $\displaystyle I = \int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$ ...(*i*)

Now numerator of integrand $= \sin^8 x - \cos^8 x$

$= (\sin^4 x)^2 - (\cos^4 x)^2$

$= (\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)$ $[\because\ a^2-b^2=(a-b)(a+b)]$

$= [(\sin^2 x)^2 - (\cos^2 x)^2]\,[(\sin^2 x)^2 + (\cos^2 x)^2]$

$= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)$ $[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x]$

$[\because\ a^2+b^2 = a^2+b^2+2ab-2ab = (a+b)^2-2ab]$

$= 1\ [-\ (\cos^2 x - \sin^2 x)]\ (1 - 2\sin^2 x \cos^2 x)$

$\Rightarrow \sin^8 x - \cos^8 x = -\cos 2x\ (1 - 2\sin^2 x \cos^2 x)$

Putting this value of $\sin^8 x - \cos^8 x$ in numerator of (i),

$$I = \int \frac{-\cos 2x\,(1-2\sin^2 x\cos^2 x)}{1-2\sin^2 x\cos^2 x}\,dx = \int -\cos 2x\ dx = -\frac{\sin 2x}{2} + c.$$

11. $\dfrac{1}{\cos(x+a)\cos(x+b)}$

Sol. Let I = $\displaystyle\int \frac{1}{\cos(x+a)\cos(x+b)}\,dx$ ...(*i*)

We know that $(x+a)-(x+b) = x+a-x-b = a-b$ ...(*ii*)

Dividing and multiplying by **sin (*a* – *b*)** in (*i*),

$$I = \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}dx$$

Replacing $(a-b)$ by $(x+a)-(x+b)$ in $\sin(a-b)$ [Using (*ii*)],

$$= \frac{1}{\sin(a-b)}\int \frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}dx$$

[∵ sin (A – B) = sin A cos B – cos A sin B]

$$= \frac{1}{\sin(a-b)}\int \left[\frac{\sin(x+a)\cos(x+b)}{\cos(x+a)\cos(x+b)} - \frac{\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]dx$$

$$\left(\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right)$$

$$= \frac{1}{\sin(a-b)}\int [\tan(x+a)-\tan(x+b)]\,dx$$

$$= \frac{1}{\sin(a-b)}[-\log|\cos(x+a)| + \log|\cos(x+b)|] + c$$

$$\left[\because \int \tan x\,dx = -\log|\cos x|\right]$$

$$= \frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right| + c. \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

Integrate the functions in Exercises 12 to 22:

12. $\dfrac{x^3}{\sqrt{1-x^8}}$

Sol. Let I = $\displaystyle\int \frac{x^3}{\sqrt{1-x^8}}\,dx = \frac{1}{4}\int \frac{4x^3}{\sqrt{1-(x^4)^2}}\,dx$ ...(*i*)

Put $x^4 = t$. Therefore $4x^3 = \frac{dt}{dx} \Rightarrow 4x^3\,dx = dt$

$\therefore$ From (i), $I = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{4}\sin^{-1} t + c$

or $I = \frac{1}{4}\sin^{-1}(x^4) + c.$

13. $\dfrac{e^x}{(1+e^x)(2+e^x)}$

Sol. Let $I = \displaystyle\int \frac{e^x}{(1+e^x)(2+e^x)}\,dx$ ...(*i*)

[Rule to evaluate $\int f(e^x)\ dx$, put $e^x = t$]

Put $e^x = t$. Therefore $e^x = \dfrac{dt}{dx}$ $\Rightarrow$ $e^x\,dx = dt$

$\therefore$ From (*i*), $I = \displaystyle\int \frac{dt}{(1+t)(2+t)} = \int \frac{1}{(t+1)(t+2)}\,dt$ ...(*ii*)

Now $t + 2 - (t + 1) = t + 2 - t - 1 = 1$

Replacing 1 in the numerator of integrand in (*ii*) by (this) $(t + 2) - (t + 1)$,

$$I = \int \frac{(t+2)-(t+1)}{(t+1)(t+2)}\,dt = \int \left(\frac{t+2}{(t+1)(t+2)} - \frac{t+1}{(t+1)(t+2)}\right) dt$$

$$= \int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt$$

$$= \log|t+1| - \log|t+2| + c = \log\left|\frac{t+1}{t+2}\right| + c$$

Putting $t = e^x$, $= \log\left|\dfrac{e^x+1}{e^x+2}\right| + c$ $= \log\left(\dfrac{e^x+1}{e^x+2}\right) + c.$

[$\because$ $e^x + 1 > 0$ and $e^x + 2 > 0$ and $|t| = t$ if $t \geq 0$]

14. $\dfrac{1}{(x^2+1)(x^2+4)}$

Sol. Let $I = \displaystyle\int \frac{1}{(x^2+1)(x^2+4)}\,dx$ ...(*i*)

Put $x^2 = y$ only in the integrand.

Now the integrand is $\dfrac{1}{(y+1)\,(y+4)}$

Let $\dfrac{1}{(y+1)\,(y+4)} = \dfrac{A}{y+1} + \dfrac{B}{y+4}$ ...(*ii*)

Multiplying by L.C.M. = $(y + 1)(y + 4)$,
$1 = A(y + 4) + B(y + 1)$
or $1 = Ay + 4A + By + B$
comparing coefficient of y, $A + B = 0$ ...(*iii*)
comparing constants, $4A + B = 1$ ...(*iv*)
Let us solve (*iii*) and (*iv*) for A and B.

(iv) − (*iii*) gives $3A = 1$ $\therefore A = \frac{1}{3}$

From (*iii*) $B = -A = -\frac{1}{3}$

Putting values of A, B and y in (*ii*),

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{\frac{1}{3}}{x^2+1} + \frac{-\frac{1}{3}}{x^2+4} = \frac{1}{3}\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right)$$

Putting this value in (*i*),

$$I = \frac{1}{3}\int\left(\frac{1}{x^2+1} - \frac{1}{x^2+2^2}\right)dx = \frac{1}{3}\left[\int\frac{1}{x^2+1}dx - \int\frac{1}{x^2+2^2}dx\right]$$

$$= \frac{1}{3}\left[\tan^{-1}x - \frac{1}{2}\tan^{-1}\frac{x}{2}\right] + c.$$

15. $\cos^3 x\, e^{\log \sin x}$

Sol. Let $I = \int \cos^3 x \; e^{\log \sin x}\, dx \quad = \int \cos^3 x \; \sin x\, dx$

$$= -\int \cos^3 x\; (-\sin x)\, dx \qquad ...(i)$$

Put $\cos x = t$. $\therefore\; -\sin x = \frac{dt}{dx} \Rightarrow -\sin x\, dx = dt$

$\therefore$ From (*i*), $I = -\int t^3\, dt = \frac{-t^4}{4} + c = \frac{-1}{4}\cos^4 x + c.$

16. $e^{3\log x}(x^4+1)^{-1}$

Sol. Let $I = \int e^{3\log x}(x^4+1)^{-1}\, dx = \int \frac{e^{\log x^3}}{x^4+1}\, dx \quad = \int \frac{x^3}{x^4+1}\, dx$

$[\because\; e^{\log f(x)} = f(x)]$

$$\Rightarrow \quad I = \frac{1}{4}\int \frac{4x^3}{x^4+1}\, dx \qquad ...(i)$$

Put $x^4 + 1 = t$. Therefore $4x^3 = \frac{dt}{dx} \Rightarrow 4x^3\, dx = dt$

$\therefore$ From (*i*), $\quad I = \frac{1}{4}\int\frac{dt}{t} = \frac{1}{4}\log|t| + c$

Putting $t = x^4 + 1$, $= \frac{1}{4}\log|x^4+1| + c \quad = \frac{1}{4}\log(x^4+1) + c.$

$[\because\; x^4 + 1 > 0 \Rightarrow |x^4+1| = x^4+1]$

17. $\int f'(ax+b)(f(ax+b))^n\, dx$

Sol. Let I = $\int f'(ax+b)\,(f(ax+b))^n \quad dx$

$$= \frac{1}{a} \int (f(ax+b))^n\, af'(ax+b) \quad dx \qquad \ldots(i)$$

Put $f(ax+b) = t$. Therefore $f'(ax+b)\,\frac{d}{dx}(ax+b) = \frac{dt}{dx}$

$\Rightarrow \; af'(ax+b)\, dx = dt$

$\therefore$ From (i), I $= \frac{1}{a}\int t^n \quad dt = \frac{1}{a}\,\frac{t^{n+1}}{n+1} + c$ if $n \neq -1$

and if $n = -1$, then $I = \frac{1}{a}\int t^{-1}\, dt = \frac{1}{a}\int \frac{1}{t}\, dt$

$$= \frac{1}{a}\log|t| + c.$$

Putting $t = f(ax + b)$, $I = \frac{(f(ax+b))^{n+1}}{a(n+1)} + c$ if $n \neq -1$

and $= \frac{1}{a}|\log f(ax + b)| + c$ if $n = -1$.

18. $\dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Sol. $I = \int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \int \frac{dx}{\sqrt{\sin^3 x(\sin x\cos\alpha + \cos x\sin\alpha)}}$

$$= \int \frac{dx}{\sqrt{\sin^3 x\,.\,\sin x(\cos\alpha + \cot x\sin\alpha)}}$$

$$= \int \frac{dx}{\sin^2 x\sqrt{\cos\alpha + \cot x\sin\alpha}} = \int \frac{\operatorname{cosec}^2 x\, dx}{\sqrt{\cos\alpha + \cot x\sin\alpha}}$$

Put $\cos\alpha + \cot x\sin\alpha = t$. Differentiating both sides

$-\operatorname{cosec}^2 x\sin\alpha\, dx = dt$

or $\operatorname{cosec}^2 x\, dx = -\frac{dt}{\sin\alpha}$

$\therefore \quad I = \int -\frac{dt}{\sin\alpha\sqrt{t}} = -\frac{1}{\sin\alpha}\int t^{-1/2}\, dt$

$$= -\frac{1}{\sin\alpha}\cdot\frac{t^{1/2}}{1/2} + c = -\frac{2}{\sin\alpha}\sqrt{\cos\alpha + \cot x\sin\alpha} + c$$

$$= -\frac{2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x}{\sin x}\sin\alpha} + c$$

$$= -\frac{2}{\sin\alpha}\sqrt{\frac{\sin x\cos\alpha + \cos x\sin\alpha}{\sin x}} + c$$

$$= -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + c.$$

19. $\dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$, $x \in [0, 1]$

Sol. We know that $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$$\therefore \quad \cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$$

$$\therefore\ I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}\, dx$$

$$= \frac{2}{\pi}\int \left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right) dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - \int 1\, dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - x + c \qquad ...(i)$$

Now let us evaluate $\int \sin^{-1}\sqrt{x}\, dx$

Put $\sqrt{x} = \sin\theta$. $\therefore\ x = \sin^2\theta$.

Differentiating both sides, $dx = 2\sin\theta\cos\theta\, d\theta = \sin 2\theta\, d\theta$

$$\therefore\ \int \sin^{-1}\sqrt{x}\, dx = \int \sin^{-1}(\sin\theta)\,.\,\sin 2\theta\, d\theta = \int \underset{I}{\theta}\ \underset{II}{\sin 2\theta}\, d\theta$$

Applying Product Rule

$$= \theta\left(\frac{-\cos 2\theta}{2}\right) - \int 1\,.\left(\frac{-\cos 2\theta}{2}\right) d\theta$$

$$= -\frac{1}{2}\theta\cos 2\theta + \frac{1}{2}\int \cos 2\theta\, d\theta = -\frac{1}{2}\theta\cos 2\theta + \frac{1}{2}\,\frac{\sin 2\theta}{2}$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{4}\, 2\sin\theta\cos\theta$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

Putting $\sin\theta = \sqrt{x}$

$$= -\frac{1}{2}(\sin^{-1}\sqrt{x})(1-2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x}$$

Putting this value of $\int \sin^{-1}\sqrt{x}\, dx$ in (i),

$$I = \frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x}\sqrt{1-x}\right] - x + c$$

$= -\frac{2}{\pi}$ ... $\sqrt{x}\sqrt{1-x}$

20. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$

(1 − 2x) − x + c.

$\sin^{-1} \underline{2}$

Sol. Let I = $\int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\ dx$

Put $\sqrt{x} = t$, *i.e.*, $\sqrt{\text{Linear}} = t$. $\therefore \quad x = t^2$

Differentiating both sides, $dx = 2t\ dt$

$\therefore \quad I = \int \sqrt{\dfrac{1-t}{1+t}}\ 2t\ dt = 2 \int t\sqrt{\dfrac{1-t}{1+t}}\ dt$

$$= 2\int t\sqrt{\frac{1-t}{1+t}\times\frac{1-t}{1-t}}\,dt \qquad \text{(Rationalising)}$$

$$= 2\int \frac{t(1-t)}{\sqrt{1-t^2}}\,dt = 2\int \frac{t-t^2}{\sqrt{1-t^2}}\,dt \qquad \ldots(i)$$

$$= 2\int \frac{(1-t^2)+t-1}{\sqrt{1-t^2}}\,dt$$

$$= 2\left[\int \sqrt{1-t^2}\,dt + \int \frac{t}{\sqrt{1-t^2}}\,dt - \int \frac{1}{\sqrt{1-t^2}}\,dt\right]$$

$$= 2\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t + \int \frac{t}{\sqrt{1-t^2}}\,dt - \sin^{-1} t\right] + c$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

or $\quad I = 2\left[\frac{1}{2}t\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t + \int \frac{t}{\sqrt{1-t^2}}\,dt\right] + c \qquad \ldots(ii)$

To evaluate $\int \frac{t}{\sqrt{1-t^2}}\,dt$

Put $1 - t^2 = z$

Differentiating both sides $-2t\,dt = dz$ or $t\,dt = -\frac{1}{2}\,dz$.

$$\therefore \int \frac{t}{\sqrt{1-t^2}}\,dt = \int \frac{-\frac{1}{2}\,dz}{\sqrt{z}} = -\frac{1}{2}\int z^{-1/2}\,dz$$

$$= -\frac{1}{2}\,\frac{z^{1/2}}{\frac{1}{2}} = -\sqrt{1-t^2} \qquad \ldots(iii)$$

Putting the value of $\int \frac{t}{\sqrt{1-t^2}}\,dt = -\sqrt{1-t^2}$ from (iii) in (ii),

We have $I = 2\left[\frac{1}{2}t \ldots - \frac{1}{2}\sin^{-1} t - \sqrt{1-t^2}\right] + c$

$\sqrt{1-t^2}$

$\lfloor 2 \quad \sqrt{1-t^2} \quad 2 \quad \sqrt{\quad} \quad \rfloor$

$$= t\sqrt{1-t^2} - \sin^{-1} t - 2\sqrt{1-t^2} + c$$

$$= (t-2)\sqrt{1-t^2} - \sin^{-1} t + c$$

Putting $t = \sqrt{x}$ $\quad = (\sqrt{x} - 2)\sqrt{1-x} - \sin^{-1}\sqrt{x} + c.$

Remark. Second method to integrate after arriving at equation **(i)** namely $I = 2\int \frac{t-t^2}{\sqrt{1-t^2}}\, dt$, is **put** $\boldsymbol{t = \sin}\ \theta$.

21. $\dfrac{2+\sin 2x}{1+\cos 2x}\, e^x$

Sol. Let I $= \int \dfrac{2+\sin 2x}{1+\cos 2x} e^x\, dx = \int e^x \dfrac{(2+2\sin x\cos x)}{2\cos^2 x}\, dx$

$= \int e^x \left(\dfrac{2}{2\cos^2 x} + \dfrac{2\sin x\cos x}{2\cos^2 x}\right) dx$

$= \int e^x \left(\dfrac{1}{\cos^2 x} + \dfrac{\sin x}{\cos x}\right) dx = \int e^x (\sec^2 x + \tan x)\, dx$

$= \int e^x (\tan x + \sec^2 x)\, dx = \int e^x (f(x) + f'(x))\, dx$

where $f(x) = \tan x$ and $f'(x) = \sec^2 x$

$= e^x f(x) + c = e^x \tan x + c.$ $\left[\because \int e^x (f(x)+f'(x))\, dx = e^x f(x) + c\right]$

22. $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$

Sol. Let I $= \int \dfrac{x^2+x+1}{(x+1)^2(x+2)}\, dx$ ...(*i*)

The integrand $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$ is a rational function of x and degree of numerator is less than degree of denominator. So we can form partial fractions of integrand.

Let integrand $\dfrac{x^2+x+1}{(x+1)^2(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2}$ ...(*ii*)

Multiplying both sides of (*ii*) L.C.M. $= (x+1)^2(x+2)$, we have

$x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

or $x^2 + x + 1 = A(x^2+3x+2) + B(x+2) + C(x^2+1+2x)$

$= Ax^2 + 3Ax + 2A + Bx + 2B + Cx^2 + C + 2Cx$

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2: $A + C = 1$ ...(*iii*)

x: $3A + B + 2C = 1$ ...(*iv*)

Constant terms: $2A + 2B + C = 1$ ...(*v*)

Let us solve Eqns. (*iii*), (*iv*) and (*v*) for A, B, C.

Eqn. (*iv*) – 2 × Eqn. (*iii*) gives (to eliminate C)

$3A + B + 2C - 2A - 2C = 1 - 2$

or $A + B = -1$ ...(*vi*)

Eqn. (*v*) – Eqn. (*iii*) gives (To eliminate C)

$$A + 2B = 0 \quad \ldots(vii)$$

Eqn. (*vii*) – Eqn. (*vi*) gives $B = 0 + 1 = 1$.

Putting $B = 1$ in (*vi*), $A + 1 = -1 \Rightarrow A = -2$

Putting $A = -2$ in (*iii*), $-2 + C = 1 \Rightarrow C = 3$

Putting values of A, B, C in (*ii*)

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$

$$\therefore \int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$$

$$= -2\int \frac{1}{x+1}\,dx + \int (x+1)^{-2}\,dx + 3\int \frac{1}{x+2}\,dx$$

$$= -2\log|x+1| + \frac{(x+1)^{-2+1}}{-2+1} + 3\log|x+2| + c$$

$$= -2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + c \quad \left(\because \frac{(x+1)^{-1}}{-1} = \frac{-1}{x+1}\right)$$

Evaluate the integrals in Exercises 23 and 24:

23. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Sol. Let $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$ ...(*i*)

Put $x = \cos 2\theta \Rightarrow \frac{dx}{d\theta} = -2\sin 2\theta$

$\Rightarrow dx = -2\sin 2\theta\, d\theta$

and $\tan^{-1}\sqrt{\frac{1-x}{1+x}} = \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}$

$= \tan^{-1}\sqrt{\tan^2\theta} = \tan^{-1}\tan\theta = \theta$

$\therefore$ From (*i*), $I = \int \theta(-2\sin 2\theta\, d\theta) = -2\int \underset{\text{I}}{\theta}\,\underset{\text{II}}{\sin 2\theta}\, d\theta$

Applying Product Rule of Integration,

$$\int \text{I.II}\,dx = \text{I}\int \text{II}\,dx - \int\left(\frac{d}{dx}(\text{I})\int \text{II}\,dx\right)dx$$

$$I = -2\left[\theta\left(\frac{-\cos 2\theta}{2}\right) - \int 1\left(\frac{-\cos 2\theta}{2}\right)d\theta\right]$$

$$= -2\left[\frac{-1}{2}\theta\cos 2\theta + \frac{1}{2}\int \cos 2\theta\, d\theta\right] = \theta\cos 2\theta - \frac{\sin 2\theta}{2} + c$$

$$= \theta\cos 2\theta - \frac{1}{2}\sqrt{1-\cos^2 2\theta} + c \quad (\because \sin^2\alpha + \cos^2\alpha = 1)$$

$$= \frac{1}{2}(\cos^{-1}x)\,x - \frac{1}{2}\sqrt{1-x^2} + c$$

$$\left[\because \cos 2\theta = x \Rightarrow 2\theta = \cos^{-1} x \Rightarrow \theta = \frac{1}{2}\cos^{-1} x\right]$$

$$= \frac{1}{2} x \cos^{-1} x - \frac{1}{2}\sqrt{1 - x^2} + c$$

$$= \frac{1}{2}\left[x \cos^{-1} x - \sqrt{1 - x^2}\right] + c.$$

24. $\dfrac{\sqrt{x^2+1}\,[\log (x^2+1) - 2\log x]}{x^4}$

Sol. $I = \displaystyle\int \frac{\sqrt{x^2+1}\,[\log (x^2+1) - 2\log x]}{x^4}\,dx$

$= \displaystyle\int \frac{\sqrt{x^2+1}}{x^4}\,[\log (x^2+1) - \log x^2]\,dx$

$= \displaystyle\int \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x^4}\,\log\left(\frac{x^2+1}{x^2}\right)dx$

$= \displaystyle\int \frac{\sqrt{1+\frac{1}{x^2}}}{x^3}\,\log\left(1+\frac{1}{x^2}\right)dx = \int \sqrt{1+\frac{1}{x^2}}\,\log\left(1+\frac{1}{x^2}\right)\cdot\frac{dx}{x^3}$

Put $1+\frac{1}{x^2} = t$ or $1 + x^{-2} = t$.

Differentiating both sides, $-\frac{2}{x^3}\,dx = dt$ or $\frac{dx}{x^3} = -\frac{1}{2}\,dt$

$\therefore\ I = -\frac{1}{2}\displaystyle\int \sqrt{t}\,\log t\,dt = -\frac{1}{2}\int \underset{\text{I}}{(\log t)}\,.\,\underset{\text{II}}{t^{1/2}}\,dt$

Integrating by Product Rule,

$= -\frac{1}{2}\left[(\log t)\,.\,\frac{t^{3/2}}{3/2} - \displaystyle\int \frac{1}{t}\,.\,\frac{t^{3/2}}{3/2}\,dt\right] = -\frac{1}{3}\,t^{3/2}\log t + \frac{1}{3}\displaystyle\int t^{1/2}\,dt$

$= -\frac{1}{3}\,t^{3/2}\log t + \frac{1}{3}\,.\,\frac{t^{3/2}}{3/2} + c$

$= \frac{2}{9}\,t^{3/2} - \frac{1}{3}\,t^{3/2}\log t + c = \frac{1}{3}\,t^{3/2}\left[\frac{2}{3} - \log t\right] + c$

Putting $t = 1 + \frac{1}{x^2}$, we have $= \frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\frac{2}{3} - \log\left(1+\frac{1}{x^2}\right)\right] + c.$

Evaluate the definite integrals in Exercises 25 to 33:

25. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$

Sol. Let I = $\displaystyle\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx = \int_{\frac{\pi}{2}}^{\pi} e^x \left[\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right] dx$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left[\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right] dx = \int_{\frac{\pi}{2}}^{\pi} e^x \left[\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right] dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left[-\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2} \right] dx \quad = \int_{\frac{\pi}{2}}^{\pi} e^x \left(f(x) + f'(x) \right) dx$$

where $f(x) = -\cot\frac{x}{2}$. Therefore $f'(x) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$

$$= \left(e^x f(x) \right)_{\frac{\pi}{2}}^{\pi} = \left(-e^x \cot\frac{x}{2} \right)_{\frac{\pi}{2}}^{\pi} \quad \left[\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) \right]$$

$$= -e^{\pi}\cot\frac{\pi}{2} - \left(-e^{\frac{\pi}{2}} \cot\frac{\pi}{4} \right)$$

$$= -e^{\pi}(0) + e^{\pi/2}(1) \quad \left[\because \cot\frac{\pi}{2} = \frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{0}{1} = 0 \right]$$

$= e^{\pi/2}$.

26. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$

Dividing every term by $\cos^4 x$,

$$I = \int_0^{\frac{\pi}{4}} \frac{\frac{\sin x \cos x}{\cos x \,.\, \cos x \,.\, \cos^2 x}}{1 + \frac{\sin^4 x}{\cos^4 x}}\, dx = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x}\, dx$$

Dividing and multiplying by 2,

$$I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{2\tan x \sec^2 x}{1 + \tan^4 x}\, dx \qquad ...(i)$$

Put $\tan^2 x = t$.

$$\therefore \quad 2\tan x \frac{d}{dx}(\tan x) = \frac{dt}{dx} \Rightarrow 2\tan x \sec^2 x\, dx = dt.$$

To change the limits of integration

When $x = 0$, $t = \tan^2 x = \tan^2 0 = 0$

When $x = \frac{\pi}{4}$, $t = \tan^2 \frac{\pi}{4} = 1$

$$\therefore \text{ From } (i),\ I = \frac{1}{2}\int_0^1 \frac{dt}{1+t^2} = \frac{1}{2}\left(\tan^{-1} t\right)_0^1$$

$$= \frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}. \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

27. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos^2 x + 4\sin^2 x}$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

Dividing every term of integrand by $\cos^2 x$,

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{(1 + 4\tan^2 x)}\, dx \qquad ...(i)$$

Put $\tan x = t$.

$\therefore \quad \sec^2 x = \dfrac{dt}{dx} \quad \Rightarrow \quad \sec^2 x\, dx = dt$

$\Rightarrow \qquad dx = \dfrac{dt}{\sec^2 x} = \dfrac{dt}{1 + \tan^2 x} = \dfrac{dt}{1 + t^2}$

To change the limits:

When $x = 0$, $t = \tan 0 = 0$

When $x = \dfrac{\pi}{2}$, $t = \tan \dfrac{\pi}{2} = \infty$

$\therefore$ From (*i*), $I = \displaystyle\int_0^{\infty} \frac{1}{1 + 4t^2} \frac{dt}{1 + t^2}$

$$= \int_0^{\infty} \frac{1}{(4t^2 + 1)(t^2 + 1)}\, dt \qquad ...(ii)$$

Put $t^2 = y$ only in the integrand of (*ii*) to form partial fractions.

The new integrand is $\dfrac{1}{(4y+1)(y+1)}$

Let $\dfrac{1}{(4y+1)(y+1)} = \dfrac{A}{4y+1} + \dfrac{B}{y+1} \qquad ...(iii)$

Multiplying by L.C.M. $= (4y + 1)(y + 1)$

$1 = A(y + 1) + B(4y + 1)$

or $1 = Ay + A + 4By + B$

Comparing coefficient of y on both sides, $A + 4B = 0 \qquad ...(iv)$

Comparing constants, $A + B = 1 \qquad ...(v)$

(*iv*) – (*v*) gives $3B = -1 \Rightarrow B = -\dfrac{1}{3}$

$\therefore$ From (*iv*) $A = -4B = -4\left(-\dfrac{1}{3}\right) = \dfrac{4}{3}$

Putting values of A, B and y in (*iii*), we have

$$\frac{1}{(4t^2+1)(t^2+1)} = \frac{\frac{4}{3}}{4t^2+1} - \frac{\frac{1}{3}}{t^2+1} = \frac{1}{3}\left(\frac{4}{(4t^2+1)} - \frac{1}{(t^2+1)}\right)$$

Putting this value in (*ii*)

$$I = \frac{1}{3}\left[4\int_0^{\infty}\frac{1}{(4t^2+1)}dt - \int_0^{\infty}\frac{1}{t^2+1}dt\right]$$

$$= \frac{1}{3}\left[4\int_0^{\infty}\frac{1}{(2t)^2+1}dt - \left(\tan^{-1}t\right)_0^{\infty}\right]$$

$$= \frac{1}{3}\left[4\left(\frac{1}{2}\tan^{-1}\frac{2t}{1}\right)_0^{\infty} - \left(\tan^{-1}t\right)_0^{\infty}\right]$$

2 → Coeff. of t

$$= \frac{1}{3}\left[2\,(\tan^{-1}\infty - \tan^{-1}0) - (\tan^{-1}\infty - \tan^{-1}0)\right]$$

$$= \frac{1}{3}\left[2.\left(\frac{\pi}{2}-0\right) - \left(\frac{\pi}{2}-0\right)\right] = \frac{1}{3}\left(\frac{2\pi}{2}-\frac{\pi}{2}\right) = \frac{1}{3}\times\frac{\pi}{2} = \frac{\pi}{6}.$$

28. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

Sol. Let $I = \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$ ...(*i*)

Put $\sin x - \cos x = t$. Differentiating both sides w.r.t.x,

$(\cos x + \sin x)\, dx = dt$

Also, squaring $\sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$

$\Rightarrow\ 1 - \sin 2x = t^2 \quad \Rightarrow\ \sin 2x = 1 - t^2$

To change the limits of Integration

When $x = \frac{\pi}{6}$, $t = \sin\frac{\pi}{6} - \cos\frac{\pi}{6}$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} = \frac{-(\sqrt{3}-1)}{2} = -\alpha \text{ (say)}$$

where $\alpha = \dfrac{\sqrt{3}-1}{2}$ ...(*ii*)

When $x = \frac{\pi}{3}$, $t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2} = \alpha$

$\therefore$ From (*i*), $I = \displaystyle\int_{-\alpha}^{\alpha}\frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1}t\right]_{-\alpha}^{\alpha}$ $= \sin^{-1}\alpha -$

$\sin^{-1}(-\alpha)$ $\rfloor_{-\alpha}$

$= \sin^{-1}\alpha + \sin^{-1}\alpha = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$. [By (*ii*)]

29. $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

Sol. Let $I = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}}\, dx$

Rationalising $= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{(\sqrt{1+x}+\sqrt{x})(\sqrt{1+x}-\sqrt{x})}\, dx$

$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x}\, dx = \int_0^1 (\sqrt{1+x}+\sqrt{x})\, dx$ $(\because\ 1+x-x=1)$

$= \int_0^1 (1+x)^{1/2}\, dx + \int_0^1 x^{1/2}\, dx = \left(\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^1 + \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^1$

$= \frac{2}{3}[(2)^{3/2}-(1)^{3/2}] + \frac{2}{3}[(1)^{3/2}-0] = \frac{2}{3}(2\sqrt{2}-1) + \frac{2}{3}(1-0)$

$\left[\because\ x^{\frac{3}{2}} = x^{\frac{2+1}{2}} = x^{1+\frac{1}{2}} = x^1 . x^{\frac{1}{2}} = x\sqrt{x}\right]$

$= \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{2}{3} = \frac{4\sqrt{2}}{3}.$

30. $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$

Sol. Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x}\, dx$

Put $\sin x - \cos x = t$. Differentiating both sides

$(\cos x + \sin x)\, dx = dt$

Also $(\sin x - \cos x)^2 = t^2$ $\therefore$ $\sin^2 x + \cos^2 x - 2\sin x\cos x = t^2$

or $1 - t^2 = \sin 2x$

Let us change the limits of Integration

When $x = 0$, $t = 0 - 1 = -1$

When $x = \frac{\pi}{4}$, $t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$\therefore$ $I = \int_{-1}^0 \frac{dt}{9 + 16(1-t^2)} = \int_{-1}^0 \frac{dt}{25 - 16t^2}$

$= \int_{-1}^0 \frac{dt}{16\left(\frac{25}{16} - t^2\right)} = \frac{1}{16}\int_{-1}^0 \frac{dt}{(5)^2}$

$$\left(16\right) \qquad \left(4\right) - t^2$$

$$= \frac{1}{16} \times \left[\frac{1}{2 \times 5/4} \log \left|\frac{5/4 + t}{5/4 - t}\right|\right]_{-1}^{0} \qquad \left[\because \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left|\frac{a + x}{a - x}\right|\right]$$

$$= \frac{1}{40}\left[\log 1 - \log \frac{1/4}{9/4}\right] = \frac{1}{40}\left[0 - \log \frac{1}{9}\right]$$

$$= \frac{1}{40}\left[-(\log 1 - \log 9)\right] = \frac{1}{40} \log 9$$

$= \frac{1}{40}\log 3^2 = \frac{2}{40}\log 3 = \frac{1}{20}\log 3.$

31. $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\, dx$

Put $\sin x = t$. Differentiating both sides $\cos x\, dx = dt$

To change the limits of Integration

When $x = 0$, $t = 0$

When $x = \frac{\pi}{2}$, $t = \sin\frac{\pi}{2} = 1$ $\quad \therefore \quad I = 2\int_0^1 t\tan^{-1} t\, dt$ ...(*i*)

Now $\int t \tan^{-1} t\, dt = \int \underset{\text{I}}{(\tan^{-1} t)}\ \underset{\text{II}}{t}\, dt$ Integrating by parts

$= \tan^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{1+t^2}\cdot\frac{t^2}{2}\, dt$

$= \frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\int \frac{(1+t^2)-1}{1+t^2}\, dt$

$= \frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\int\left(1 - \frac{1}{1+t^2}\right) dt = \frac{t^2}{2}\tan^{-1} t - \frac{1}{2}(t - \tan^{-1} t)$

$= \frac{t^2}{2}\tan^{-1} t - \frac{1}{2}t + \frac{1}{2}\tan^{-1} t + c = \frac{1}{2}[(t^2+1)\tan^{-1} t - t]$

From (*i*), $I = 2\left[\frac{1}{2}\{(t^2+1)\tan^{-1} t - t\}\right]_0^1 = (2\tan^{-1} 1 - 1) - (0 - 0)$

$= 2 \times \frac{\pi}{4} - 1 = \frac{\pi}{2} - 1.$

32. $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\, dx$

Sol. Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\, dx = \int_0^{\pi} \frac{x\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}\, dx$

cos x cos x

$$= \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx \qquad \ldots(i)$$

Using $\int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx$

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)}\, dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \sin x}\, dx \qquad \ldots(ii)$$

Adding Eqns. (i) and (ii), we have

$$2I = \int_0^{\pi} \frac{x \sin x + (\pi - x)\sin x}{1 + \sin x}\, dx = \int_0^{\pi} \frac{x \sin x + \pi \sin x - x \sin x}{1 + \sin x}\, dx$$

$$= \int_0^\pi \frac{\pi \sin x}{1+\sin x}\,dx = \pi \int_0^\pi \frac{\sin x}{1+\sin x}\,dx$$

or $\quad 2I = \pi \int_0^\pi \frac{(1+\sin x)-1}{1+\sin x}\,dx$

$$\Rightarrow \quad 2I = \pi \int_0^\pi \left(1 - \frac{1}{1+\sin x}\right) dx = \pi \int_0^\pi dx - \pi \int_0^\pi \frac{dx}{1+\sin x}$$

$$= \pi \left[x\right]_0^\pi - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$$

$$= \pi(\pi) - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin\left(\frac{\pi}{2}-x\right)} = \pi^2 - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$$

$$= \pi^2 - 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{2\cos^2 \frac{x}{2}} = \pi^2 - \pi \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\,dx$$

or $\quad 2I = \pi^2 - \pi \left[\dfrac{\tan \frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} = \pi^2 - 2\pi(1)$

Dividing both sides by 2, $I = \dfrac{\pi^2}{2} - \pi = \pi\left(\dfrac{\pi}{2} - 1\right) = \pi\left(\dfrac{\pi-2}{2}\right)$.

33. $\int_1^4 [|x-1|+|x-2|+|x-3|]\,dx$

Sol. Let $I = \int_1^4 (|x-1|+|x-2|+|x-3|)\,dx \quad \ldots(i)$

Putting each expression within modulus equal to 0, we have

$x - 1 = 0,\ x - 2 = 0,\ x - 3 = 0$ *i.e.,* $x = 1,\ x = 2,\ x = 3$

Here 2 and $3 \in (1, 4)$

$\therefore$ From (*i*), $I = \int^2 (|x-1|+|x-2|+|x-3|)\,dx$

$$+ \int_{2}^{3} (|x-1|+|x-2|+|x-3|)\ dx + \int_{3}^{4} (|x-1|+|x-2|+|x-3|)\ dx \qquad ...(ii)$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx \text{ where } a < c < d < b\right]$$

Let $I_1 = \int_{1}^{2} (|x-1|+|x-2|+|x-3|)\ dx$

On this interval (1, 2) (for example taking $x = 1.3$; $(x - 1)$ is positive, $(x - 2)$ is negative and $(x - 3)$ is negative and hence $|x - 1| = (x - 1)$, $|x - 2| = -(x - 2)$ and $|x - 3| = -(x - 3)$).

Therefore $I_1 = \int_1^2 ((x-1)-(x-2)-(x-3))\,dx$

$= \int_1^2 (x-1-x+2-x+3)\,dx = \int_1^2 (4-x)\,dx$

$= \left(4x - \frac{x^2}{2}\right)_1^2 = (8-2) - \left(4 - \frac{1}{2}\right)$

$= 6 - 4 + \frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2}$ ...(*iii*)

Let $I_2 = \int_2^3 (|x-1|+|x-2|+|x-3|)\,dx$

On this interval (2, 3) (for example taking $x = 2.8$; $(x-1)$ is positive, $(x-2)$ is positive and $(x-3)$ is negative and hence $|x-1| = x-1$, $|x-2| = x-2$ and $|x-3| = -(x-3)$)

Therefore $I_2 = \int_2^3 ((x-1+x-2-(x-3))\,dx = \int_2^3 (2x-3-x+3)\,dx$

$= \int_2^3 x\,dx = \left(\frac{x^2}{2}\right)_2^3 = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$ ...(*iv*)

Let $I_3 = \int_3^4 (|x-1|+|x-2|+|x-3|)\,dx$

On this interval (3, 4), (for example taking $x = 3.4$; $(x-1)$ is positive, $(x-2)$ is positive and $(x-3)$ is positive and hence $|x-1| = x-1$, $|x-2| = x-2$ and $|x-3| = x-3$))

Therefore $I_3 = \int_3^4 (x-1+x-2+x-3)\,dx = \int_3^4 (3x-6)\,dx$

$= \left(\frac{3x^2}{2} - 6x\right)_3^4 = (24-24) - \left(\frac{27}{2} - 18\right)$

$= 0 - \left(\frac{27-36}{2}\right) = -\left(-\frac{9}{2}\right) = \frac{9}{2}$ ...(*v*)

Putting values of I_1, I_2, I_3 from (*iii*), (*iv*) and (*v*) in (*ii*),

$$I = \frac{5}{2} + \frac{5}{2} + \frac{9}{2} = \frac{19}{2}.$$

Prove the following (Exercises 34 to 40):

34. $\int^3 \frac{dx}{\quad} = \frac{2}{} + \log \frac{2}{}$

1 x^2 (x + 1) 3 3

Sol. Let I = $\int_1^3 \frac{dx}{x^2(x+1)} = \int_1^3 \frac{1}{x^2(x+1)}\, dx$ ...(*i*)

Let integrand $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ ...(*ii*)

(Partial fractions)

Multiplying by L.C.M. = $x^2\,(x + 1)$

$1 = Ax(x + 1) + B(x + 1) + Cx^2$

$\Rightarrow$ 1 = Ax^2 + Ax + Bx + B + Cx^2

Comparing coefficients of x^2, x and constant terms on both sides, we have

x^2: A + C = 0 ...(*iii*)

x: A + B = 0 ...(*iv*)

Constants: B = 1 ...(*v*)

Let us solve (*iii*), (*iv*), (*v*) for A, B, C.

Putting B = 1 from (*v*) in (*iv*), A + 1 = 0 or A = – 1

Putting A = – 1 in (*iii*), – 1 + C = 0 $\Rightarrow$ C = 1

Putting values of A, B, C in (*ii*),

$$\frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$\therefore$ From (*i*), I = $\int_1^3 \frac{dx}{x^2(x+1)}$

$$= -\int_1^3 \frac{1}{x}\,dx + \int_1^3 \frac{1}{x^2}\,dx + \int_1^3 \frac{1}{x+1}\,dx$$

$$= -\left(\log|x|\right)_1^3 + \int_1^3 x^{-2}\,dx + \left(\log|x+1|\right)_1^3$$

$$= -(\log|3| - \log|1|) + \left(\frac{x^{-1}}{-1}\right)_1^3 + (\log|4| - \log|2|)$$

$$= -\log 3 + 0 - \left(\frac{1}{x}\right)_1^3 + \log 4 - \log 2$$

$$= -\log 3 - \left(\frac{1}{3} - 1\right) + \log 2^2 - \log 2$$

$$= -\log 3 - \left(\frac{1-3}{3}\right) + 2\log 2 - \log 2$$

$$= -\log 3 + \frac{2}{3} + \log 2 = \frac{2}{3} + \log 2 - \log 3$$

$$= \frac{2}{3} + \log\frac{2}{3}.$$

35. $\int_0^1 x\,e^x\,dx = 1$

Sol. $\int_0^1 x\,e^x$

I II

Applying Product Rule of definite Integration

$$\int_a^b \text{I}\,.\,\text{II}\,dx = \left(\text{I}\int \text{II}\,dx\right)_a^b - \int_a^b \left(\frac{d}{dx}(\text{I})\int \text{II}\,dx\right)dx$$

$$= \left(x\,e^x\right)_0^1 - \int_0^1 1.\,e^x\,dx$$

$$= e - 0 - \int_0^1 e^x\,dx = e - \left(e^x\right)_0^1$$

$$= e - (e - e^0) = e - e + e^0 = 1.$$

36. $\int_{-1}^{1} x^{17} \cos^4 x \, dx = 0$

Sol. Let I = $\int_{-1}^{1} x^{17} \cos^4 x \, dx$ ...(*i*)

Here the integrand $f(x) = x^{17} \cos^4 x$

$\therefore \quad f(-x) = (-x)^{17} \cos^4 (-x)$

$= -x^{17} \cos^4 x = -f(x)$

$\therefore$ $f(x)$ is an odd function of x.

$\therefore$ From (*i*), I = $\int_{-1}^{1} x^{17} \cos^4 x \, dx = 0$

[$\because$ If $f(x)$ is an odd function of x, then $\int_{-a}^{a} f(x) \, dx = 0$]

37. $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \frac{2}{3}$

Sol. $\int_0^{\frac{\pi}{2}} \sin^3 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1}{4} (3 \sin x - \sin 3x) \, dx$

$\left[\because \sin 3A = 3 \sin A - 4 \sin^3 A \Rightarrow \sin^3 A = \frac{1}{4}(3 \sin A - \sin 3A)\right]$

$= \frac{1}{4}\left[3(-\cos x) - \left(-\frac{\cos 3x}{3}\right)\right]_0^{\pi/2} = \frac{1}{4}\left(-3 \cos x + \frac{1}{3} \cos 3x\right)_0^{\pi/2}$

$= \frac{1}{4}\left[\left(-3 \cos \frac{\pi}{2} + \frac{1}{3} \cos \frac{3\pi}{2}\right) - \left(-3 \cos 0 + \frac{1}{3} \cos 0\right)\right]$

$= \frac{1}{4}\left[-3 \times 0 + \frac{1}{3} \times 0 + 3 \times 1 - \frac{1}{3} \times 1\right] = \frac{1}{4}\left(3 - \frac{1}{3}\right)$

$= \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}.$

$\left[\because \cos \frac{3\pi}{2} = \cos 270° = \cos(180° + 90°) = -\cos 90° = 0\right]$

38. $\int_0^{\frac{\pi}{4}} 2 \tan^3 x \, dx = 1 - \log 2$

Sol. Let $I = \int_0^{\frac{\pi}{4}} 2\tan^3 x \, dx = 2\int_0^{\frac{\pi}{4}} \tan x \cdot \tan^2 x \, dx$

Replacing $\tan^2 x$ by $(\sec^2 x - 1)$ in the integrand,

$$I = 2\int_0^{\frac{\pi}{4}} \tan x (\sec^2 x - 1) \, dx = 2\left[\int_0^{\frac{\pi}{4}} (\tan x \sec^2 x - \tan x) \, dx\right]$$

$$= 2\left[\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan x \, dx\right] \quad \ldots(i)$$

Let $I_1 = \int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$

Put tan x = t. Therefore $\sec^2 x = \frac{dt}{dx}$ $\therefore$ $\sec^2 x\, dx = dt$

To change the limits of Integration

When $x = 0$, $t = \tan x = \tan 0 = 0$

When $x = \frac{\pi}{4}$, $t = \tan\frac{\pi}{4} = 1$

$\therefore\ I_1 = \int_0^1 t\ dt = \left(\frac{t^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

Putting this value of I_1 in (i),

$$I = 2\left[\frac{1}{2} - \left(\log \sec x\right)_0^{\pi/4}\right] = 1 - 2\left(\log\sec\frac{\pi}{4} - \log\sec 0\right)$$

$$= 1 - 2(\log\sqrt{2} - \log 1) = 1 - 2(\log 2^{1/2} - 0)$$

$$= 1 - 2\left(\frac{1}{2}\log 2\right) = 1 - \log 2.$$

39. $\int_0^1 \sin^{-1}x\ dx = \frac{\pi}{2} - 1$

Sol. Put x = sin θ. Differentiating both sides $dx = \cos\theta\ d\theta$

To change the limits of Integration

When $x = 0$, $\theta = 0$,

When $x = 1$, $\sin\theta = 1$ and therefore $\theta = \frac{\pi}{2}$

$\therefore\ \int_0^1 \sin^{-1} x\ dx = \int_0^{\frac{\pi}{2}} \underset{\text{I}}{\theta}\, \underset{\text{II}}{\cos\theta}\ d\theta$

Integrating by parts

$$= \left[\theta\sin\theta\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} 1.\sin\theta\ d\theta = \left(\frac{\pi}{2} - 0\right) + \left[\cos\theta\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \left(\cos\frac{\pi}{2} - \cos 0\right) = \frac{\pi}{2} + (0 - 1) = \frac{\pi}{2} - 1.$$

40. Evaluate $\int_0^1 e^{2-3x}\,dx$ as a limit of a sum.

Sol. Step I. Comparing $\int_0^1 e^{2-3x}\ dx$ with $\int_a^b f(x)\ dx$, we have

$a = 0$, $b = 1$, $f(x) = e^{2-3x}$

$\therefore\ nh = b - a = 1$

Step II. Putting $x = a, a + h, a + 2h, a + (n - 1)\ h$ in $f(x)$, we have

$f(a) = f(0) = e^2$

$f(a + h) = f(h) = e^{2-3h}$

$f(a + 2h) = f(2h) = e^{2-6h}$

$f(a + (n - 1)h) = f((n - 1)\ h) = e^{2-3(n-1)\ h}$

Step III. Putting these values in

$$\int_a^b f(x) = \lim_{\substack{n\to\infty \\ h\to 0}} h[f(a)+f(a+h)+f(a+2h)+\ldots + f\{a+(n-1)h\}],$$

we have $\int_0^1 e^{2-3x}\, dx = \lim_{\substack{n\to\infty \\ h\to 0}} h[e^2 + e^{2-3h} + e^{2-6h} + \ldots + e^{2-3(n-1)h}]$

$$= \lim_{\substack{h\to 0 \\ n\to\infty}} h \cdot e^2 [1 + e^{-3h} + e^{-6h} + \ldots + e^{-3(n-1)h}]$$

[The series within brackets is a G.P. series of n terms with $a = 1$, $r = e^{-3h}$ and using S_n of G.P. $= a\dfrac{(r^n - 1)}{r-1}$]

$$= e^2 \lim_{\substack{h\to 0 \\ n\to\infty}} h \cdot \left[\frac{e^{-3nh}-1}{e^{-3h}-1}\right] \qquad \left[\because (e^{-3h})^n = e^{-3nh}\right]$$

Step IV. Putting $nh = 1$

$$= e^2 \lim_{h\to 0} h \cdot \left[\frac{e^{-3}-1}{e^{-3h}-1}\right]$$

Step V. Taking limits as $h \to 0$,

$$= e^2 (e^{-3} - 1) \lim_{h\to 0} \frac{-3h}{e^{-3h}-1} \times \left(-\frac{1}{3}\right)$$

$$= (e^{-1} - e^2) \times 1 \times \left(-\frac{1}{3}\right) \qquad \left[\because \lim_{x\to 0}\frac{x}{e^x-1} = 1\right]$$

$$= \frac{1}{3}\left(e^2 - \frac{1}{e}\right)$$

41. Choose the correct answer: $\int \dfrac{dx}{e^x + e^{-x}}$ **is equal to**

(A) $\tan^{-1}(e^x) + c$ **(B)** $\tan^{-1}(e^{-x}) + c$

(C) $\log(e^x - e^{-x}) + c$ **(D)** $\log(e^x + e^{-x}) + c$

Sol. Let I $= \int \dfrac{dx}{\ } = \dfrac{1}{\ } dx$

$$e^x + e^{-x} \qquad \int e^x + \left(\frac{1}{e^x}\right)$$

$$= \int \frac{1}{\left(\frac{e^{2x}+1}{e^x}\right)}\, dx = \int \frac{e^x}{e^{2x}+1}\, dx \qquad \ldots(i)$$

$$[\because\ e^x \cdot e^x = e^{x+x} = e^{2x}]$$

Put $e^x = t$. $\left[\because \text{ For } \int f(e^x)\,dx, \text{ put } e^x = t\right]$

Therefore $e^x = \frac{dt}{dx}$. Therefore $e^x\,dx = dt$

$\therefore$ From (*i*), $I = \int \frac{dt}{t^2+1} = \tan^{-1} t + c$

$= \tan^{-1}(e^x) + c$

$\therefore$ Option (A) is the correct answer.

42. Choose the correct answer:

$$\int \frac{\cos 2x}{(\sin x + \cos x)^2}\,dx \text{ is equal to}$$

(A) $\frac{-1}{\sin x + \cos x} + c$ (B) $\log|\sin x + \cos x| + c$

(C) $\log|\sin x - \cos x| + c$ (D) $\frac{1}{(\sin x + \cos x)^2}$.

Sol. Let $I = \int \frac{\cos 2x}{(\sin x + \cos x)^2}\,dx = \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2}\,dx$

$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\sin x + \cos x)(\sin x + \cos x)}\,dx = \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$

$= \log|\sin x + \cos x| + c.$ $\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$

OR

Put denominator $\sin x + \cos x = t$.

$\therefore$ Option (B) is the correct answer.

43. Choose the correct answer:

If $f(a + b - x) = f(x)$, then $\int_a^b x\,f(x)\,dx$ is equal to

(A) $\frac{a+b}{2}\int_a^b f(b-x)\,dx$ (B) $\frac{a+b}{2}\int_a^b f(b+x)\,dx$

(C) $\frac{b-a}{2}\int_a^b f(x)\,dx$ (D) $\frac{a+b}{2}\int_a^b f(x)\,dx$.

2 $\int_a^b$ **2** ***a***

Sol. **Given:** $f(a + b - x) = f(x)$ ...(*i*)

Let $I = \int_a^b x\, f(x)\, dx$ ...(*ii*)

Changing x to $(a + b - x)$ in the Integrand on Right side (*ii*).

$$I = \int_a^b (a + b - x)\, f(a + b - x)\, dx \quad ...(iii)$$

$$\left[\because \text{ By Property of definite integrals, } \int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx\right]$$

Putting $f(a+b-x) = f(x)$ from (i) in integrand of (iii),

$$I = \int_a^b f(a+b-x)\, f(x)\, dx \qquad ...(iv)$$

Adding (ii) and (iv), we have $2I = \int_a^b [x\, f(x) + (a+b-x)\, f(x)]\, dx$

$$2I = \int_a^b (x+a+b-x)\, f(x)\, dx = \int_a^b (a+b)\, f(x)\, dx = (a+b)\int_a^b f(x)\, dx$$

Dividing by 2, $I = \left(\frac{a+b}{2}\right)\int_a^b f(x)\, dx$

or $\int_a^b x\, f(x)\, dx = \left(\frac{a+b}{2}\right)\int_a^b f(x)\, dx$

$\therefore$ Option (D) is the correct answer.

44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ is

(A) 1 **(B) 0** **(C) – 1** **(D) $\frac{\pi}{4}$**

Sol. Let $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx = \int_0^1 \tan^{-1}\left(\frac{x+x-1}{1-x^2+x}\right) dx$

$$= \int_0^1 \tan^{-1}\left(\frac{x+(x-1)}{1-x(x-1)}\right) dx = \int_0^1 (\tan^{-1} x + \tan^{-1}(x-1))\, dx$$

$$\left[\because \tan^{-1}\frac{x+y}{1-xy} = \tan^{-1} x + \tan^{-1} y\right]$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(x-1)\, dx$$

Changing x to $(1-x)$ in integrand of second integral

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x-1)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(-x)\, dx = \int_0^1 \tan^{-1} x\, dx - \int_0^1 \tan^{-1} x\, dx = 0.$$

[$\because$ $\tan^{-1}(-x) = -\tan^{-1}x$]

$\therefore$ Option (B) is the correct answer.