Exercise 5.1

1. **Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.**

Sol. Given: $f(x) = 5x - 3$ ...(*i*)

Continuity at $x = 0$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (5x - 3)$ (By (*i*))

Putting $x = 0$, $= 5(0) - 3 = 0 - 3 = -3$

Putting $x = 0$ in (*i*), $f(0) = 5(0) - 3 = -3$

$\therefore \lim_{x \to 0} f(x) = f(0)\ (= -3)$ $\therefore$ $f(x)$ is continuous at $x = 0$.

Continuity at $x = -3$

$\lim_{x \to -3} f(x) = \lim_{x \to -3} (5x - 3)$ (By (*i*))

Putting $x = -3$, $= 5(-3) - 3 = -15 - 3 = -18$

Putting $x = -3$ in (*i*), $f(-3) = 5(-3) - 3 = -15 - 3 = -18$

$\therefore \lim_{x \to -3} f(x) = f(-3)(= -18)$

$\therefore$ $f(x)$ is continuous at $x = -3$.

Continuity at $x = 5$

$\lim_{x \to 5} f(x) = \lim_{x \to 5} (5x - 3)$ (By (*i*))

Putting $x = 5$, $5(5) - 3 = 25 - 3 = 22$

Putting $x = 5$ in (*i*), $f(5) = 5(5) - 3 = 25 - 3 = 22$

$\therefore \lim_{x \to 5} (5x - 3) = f(5)\ (= 22)$ $\therefore$ $f(x)$ is continuous at $x = 5$.

2. Examine the continuity of the function
$f(x) = 2x^2 - 1$ at $x = 3$.

Sol. Given: $f(x) = 2x^2 - 1$ ...(*i*)

Continuity at $x = 3$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x^2 - 1) \quad \text{[By } (i)]$$

Putting $x = 3$, $= 2.3^2 - 1 = 2(9) - 1 = 18 - 1 = 17$
Putting $x = 3$ in (*i*), $f(3) = 2.3^2 - 1 = 18 - 1 = 17$

$\therefore \lim_{x \to 3} f(x) = f(3)\ (= 17)$ $\therefore$ $f(x)$ is continuous at $x = 3$.

3. Examine the following functions for continuity:

(*a*) $f(x) = x - 5$ **(*b*) $f(x) = \dfrac{1}{x - 5}$, $x \neq 5$**

(c) $f(x) = \dfrac{x^2 - 25}{x + 5}$, $x \neq -5$ **(*d*) $f(x) = |x - 5|$.**

Sol. (*a*) **Given:** $f(x) = x - 5$ ...(*i*)

The domain of f is R
($\because$ $f(x)$ is real and finite for all $x \in$ R)
Let c be any real number (*i.e.,* $c \in$ domain of f).

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) \quad \text{[By } (i)]$$

Putting $x = c$, $= c - 5$

Putting $x = c$ in (*i*), $f(c) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)\ (= c - 5)$

$\therefore$ f is continuous at every point c in its domain (here R).
Hence f is continuous.

Or

Here $f(x) = x - 5$ is a polynomial function. We know that every polynomial function is continuous (see note below).
Hence $f(x)$ is continuous (in its domain R)
Very important Note. The following functions are continuous (for all x in their domain).
1. Constant function
2. Polynomial function.
3. Rational function $\dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions of x and $g(x) \neq 0$.
4. Sine function ($\Rightarrow$ sin x).
5. cos x.
6. e^x.
7. e^{-x}.
8. log x ($x > 0$).

9. Modulus function.

(*b*) **Given:** $f(x) = \frac{1}{x-5}, x \neq 5$ ...(*i*)

Given: The domain f is $R - (x \neq 5)$ *i.e.,* $R - \{5\}$

($\because$ For $x = 5$, $f(x) = \frac{1}{x-5} = \frac{1}{5-5} = \frac{1}{0} \to \infty$

$\therefore$ $5 \notin$ domain of f)

Let c be any real number such that $c \neq 5$

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x-5} \quad \text{[By } (i)]$$

Putting $x = c$, $= \frac{1}{c-5}$

Putting $x = c$ in (i), $f(c) = \frac{1}{c-5}$

$\therefore$ $\lim_{x \to c} f(x) = f(c) \left(= \frac{1}{c-5} \right)$

$\therefore$ $f(x)$ is continuous at every point c in the domain of f.

Hence f is continuous.

Or

Here $f(x) = \frac{1}{x-5}$, $x \neq 5$ is a rational function $\left(= \frac{\text{Polynomial 1 of degree 0}}{\text{Polynomial } (x-5) \text{ of degree 1}} \right)$ and its denominator *i.e.,* $(x - 5) \neq 0$ ($\because$ $x \neq 5$). We know that every rational function is continuous (By Note below Solution of Q. No. 3(a)). Therefore f is continuous (in its domain R – {5}).

(c) $f(x) = \frac{x^2 - 25}{x+5}$, $x \neq -5$

Here $f(x) = \frac{x^2 - 25}{x+5}$, $x \neq -5$ is a rational function and denominator $x + 5 \neq 0$ ($\because$ $x \neq -5$).

(In fact $f(x) = \frac{x^2 - 25}{x+5}$, $(x \neq -5) = \frac{(x+5)(x-5)}{x+5}$ $= x - 5$, $(x \neq -5)$ is a polynomial function). We know thatevery rational function is continuous. Therefore f is continuous (in its domain R – {– 5}).

Or

Proceed as in Method I of Q. No. 3(b).

(d) **Given:** $f(x) = |x - 5|$

Domain of $f(x)$ is R ($\because$ $f(x)$ is real and finite for all real x in $(-\infty, \infty)$)

Here $f(x) = |x - 5|$ is a modulus function.

We know that every modulus function is continuous.

(By Note below Solution of Q. No. 3(a)). Therefore f is continuous in its domain R.

4. Prove that the function $f(x) = x^n$ is continuous at $x = n$ where n is a positive integer.

Sol. Given: $f(x) = x^n$ where n is a positive integer ....(i)

Domain of $f(x)$ is R (....$f(x)$ is real and finite for all real x)

Here $f(x) = x^n$, where n is a positive integer.

We know that every polynomial function of x is a continuous function. Therefore, f is continuous (in its whole domain R) and hence continuous at $x = n$ also.

Or

$$\lim_{x \to n} f(x) = \lim_{x \to n} x^n \qquad \text{[By (i)]}$$

Putting $x = n$, $= n^n$

Again putting $x = n$ in (i), $f(n) = n^n$

$\therefore \lim_{x \to n} f(x) = f(n)\ (= n^n)$ $\qquad \therefore f(x)$ is continuous at $x = n$.

5. Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$?, At $x = 1$?, At $x = 2$?

Sol. Given: $f(x) = \begin{cases} x, & \text{if } x \le 1 & ...(i) \\ 5, & \text{if } x > 1 & ...(ii) \end{cases}$

(Read Note **(on continuity)** before the solution of Q. No. 1 of this exercise)

Continuity at $x = 0$

Left Hand Limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x$ [By (i)]

$(x \to 0^- \Rightarrow x <$ slightly less than $0 \Rightarrow x < 1)$

Putting $x = 0$, $= 0$

Right hand limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x$ [By (i)]

$(x \to 0^+ \Rightarrow x$ is slightly greater than 0 say $x = 0.001 \Rightarrow x < 1)$

Putting $x = 0$, $\lim_{x \to 0^+} f(x) = 0$ $\qquad \therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$

$\therefore \lim_{x \to 0} f(x)$ exists and $= 0 = f(0)$

$(\because$ Putting $x = 0$ in (i), $f(0) = 0)$

$\therefore f(x)$ is continuous at $x = 0$.

Continuity at $x = 1$

Left Hand Limit $= \lim\limits_{x \to \ ^{-}} f(x) = \lim\limits_{x \to \ ^{-}} x$ [By (i)]

Putting $x = 1$, $= 1$

Right Hand Limit $= \lim\limits_{x \to 1^{+}} f(x) = \lim\limits_{x \to 1^{+}} 5$

Putting $x = 1$, $\lim\limits_{x \to 1^{+}} f(x) = 5$

$\therefore \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$ $\therefore \lim\limits_{x \to 1} f(x)$ does not exist.

$\therefore$ $f(x)$ is discontinuous at $x = 1$.

Continuity at $x = 2$

Left Hand Limit $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 5$ [By (ii)]

($x \to 2 - \Rightarrow x$ is slightly $< 2 \Rightarrow x = 1.98$ (say) $\Rightarrow x > 1$)

Putting $x = 2$, $= 5$

Right Hand Limit $= \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 5$ [By (ii)]

($x \to 2 + \Rightarrow x$ is slightly > 2 and hence $x > 1$ also)

Putting $x = 2$, $= 5$

$\therefore \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$ $(= 5)$

$\therefore \lim\limits_{x \to 2} f(x)$ exists and $= 5 = f(2)$

(Putting $x = 2 > 1$ in (ii), $f(2) = 5$)

$\therefore$ $f(x)$ is continuous at $x = 2$

Answer. f is continuous at $x = 0$ and $x = 2$ but not continuous at $x = 1$.

Find all points of discontinuity of f, where f is defined by (Exercises 6 to 12)

6. $f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ 2 - & x > 2 \end{cases}$.

Sol. Given: $f(x) = 2x + 3$, $x \leq 2$ ...(i)

$= 2x - 3$ $x > 2$ ...(ii)

To find points of discontinuity of f (in its domain)

Here $f(x)$ is defined for $x \leq 2$ *i.e.,* on $(-\infty, 2]$

and also for $x > 2$ *i.e.,* on $(2, \infty)$

$\therefore$ Domain of f is $(-\infty, 2] \cup (2, \infty) = (-\infty, \infty) = \mathbb{R}$

By (i), for all $x < 2$ ($x = 2$ being partitioning point can't be mentioned here) $f(x) = 2x + 3$ is a polynomial and hence continuous.

By (ii), for all $x > 2$, $f(x) = 2x - 3$ is a polynomial and hence continuous.

Therefore $f(x)$ is continuous on $\mathbb{R} - \{2\}$.

Let us examine continuity of f at partitioning point $x = 2$

Left Hand Limit $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2x + 3)$ [By (i)]

Putting $x = 2$, $= 2(2) + 3 = 4 + 3 = 7$

Right Hand Limit = $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3)$ [By (*ii*)]

Putting $x = 2$, $= 2(2) - 3 = 4 - 3 = 1$

$\therefore \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

$\therefore \lim_{x \to 2} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 2$ (only).

7. $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 & ...(i) \\ -2x, & \text{if } -3 < x < 3 & ...(ii) \\ 6x + 2, & \text{if } x \ge 3 & ...(iii) \end{cases}$

Here $f(x)$ is defined for $x \le -3$ *i.e.,* $(-\infty, -3]$ and also for $-3 < x < 3$ and also for $x \ge 3$ *i.e.,* on $[3, \infty)$.

$\therefore$ Domain of f is $(-\infty, -3] \cup (-3, 3) \cup [3, \infty) = (-\infty, \infty) = R$.

By (i), for all $x < -3$, $f(x) = |x| + 3 = -x + 3$

($\because x < -3$ means x is negative and hence $|x| = -x$)

is a polynomial and hence continuous.

By (ii), for all x $(-3 < x < 3)$ $f(x) = -2x$ is a polynomial and hence continuous.

By (iii), for all $x > 3$, $f(x) = 6x + 2$ is a polynomial and hence continuous. Therefore, $f(x)$ is continuous on $R - \{-3, 3\}$.

From (i), (ii) and (iii) we can observe that $x = -3$ and $x = 3$ are partitioning points of the domain R.

Let us examine continuity of f at partitioning point $x = -3$

Left Hand Limit $= \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (|x| + 3)$ [By (i)]

($\because x \to -3^- \Rightarrow x < -3$)

$= \lim_{x \to -3^-} (-x + 3)$

($\because x \to -3^- \Rightarrow x < -3$ means x is negative and hence $|x| = -x$)

Put $x = -3$, $= 3 + 3 = 6$

Right Hand Limit $= \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x)$ [By (ii)]

($\because x \to -3^+ \Rightarrow x > -3$)

Putting $x = -3$, $= -2(-3) = 6$

$\therefore \lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x)$ $(= 6)$

$\therefore \lim_{x \to -3} f(x)$ exists and $= 6$

Putting $x = -3$ in (i), $f(-3) = |-3| + 3 = 3 + 3 = 6$

$\therefore \lim_{x \to -3} f(x) = f(-3) \; (= 6)$

$\therefore$ $f(x)$ is continuous at $x = -3$.

Now let us examine continuity of f at partitioning point $x = 3$

Left Hand Limit = $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x)$ [By (*ii*)]

$(\because\ x \to 3^- \Rightarrow x < 3)$

Putting $x = 3$, $= -2(3) = -6$

Right Hand Limit = $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$ [By (*iii*)]

$(\because\ x \to 3^+ \Rightarrow x > 3)$

Putting $x = 3$, $= 6(3) + 2 = 18 + 2 = 20$

$\therefore \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore \lim_{x \to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 3$ (only).

8. $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

Sol. Given: $f(x) = \dfrac{|x|}{x}$ if $x \neq 0$

[*i.e.,* $= \dfrac{x}{x} = 1$ if $x > 0$ $(\because$ For $x > 0$, $|x| = x)$

and $= -\dfrac{x}{x} = -1$ if $x < 0$ $(\because$ For $x < 0$, $|x| = -x)$

i.e., $f(x) = 1$ if $x > 0$ ...(*i*)

$= -1$ if $x < 0$ ...(*ii*)

$= 0$ if $x = 0$ ...(*iii*)

Clearly domain of $f(x)$ is R $(\because\ f(x)$ is defined for $x > 0$, for $x < 0$ and also for $x = 0)$

By (*i*), for all $x > 0$, $f(x) = 1$ is a constant function and hence continuous.

By (*ii*), for all $x < 0$, $f(x) = -1$ is a constant function and hence continuous.

Therefore $f(x)$ is continuous on R – {0}.

Let us examine continuity of f at the partitioning point $x = 0$

Left Hand Limit = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1$ [By (*ii*)]

$(\because\ x \to 0^- \Rightarrow x < 0)$

Put $x = 0$, $= -1$

Right Hand Limit = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1$ [By (*i*)]

$(\because\ x \to 0^+ \Rightarrow x > 0)$

Put $x = 0$, $= 1$

$\therefore \quad \lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$

$\therefore \quad \lim\limits_{x \to 0} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 0$ (only).

Note. It may be noted that the function given in Q. No. 8 is called a **signum function.**

9. $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$.

Sol. Given:

$$f(x) = \begin{cases} \dfrac{x}{|x|}, \text{ if } x < 0 = \dfrac{x}{-x} = -1 \text{ if } x < 0 & \ldots(i) \\ \qquad (\because \text{ For } x < 0, |x| = -x) \\ -1 \quad \text{if } x \geq 0 & \ldots(ii) \end{cases}$$

Here $f(x)$ is defined for $x < 0$ *i.e.,* on $(-\infty, 0)$ and also for $x \geq 0$ *i.e.,* on $[0, \infty)$.

$\therefore$ Domain of f is $(-\infty, 0) \cup [0, \infty) = (-\infty, \infty) = R$.

From (i) and (ii), we find that

$f(x) = -1$ for all real x (< 0 as well as ≥ 0)

Here $f(x) = -1$ is a constant function.

We know that every constant function is continuous.

$\therefore f$ is continuous (for all real x in its domain R)

Hence no point of discontinuity.

10. $f(x) = \begin{cases} x + 1, & \text{if } x \geq 1 \\ ^2 + 1, & \text{if } x < 1 \end{cases}$.

Sol. Given: $\begin{cases} x + 1, & \text{if } x \geq 1 & \ldots(i) \\ x^2 + 1, & \text{if } x < 1 & \ldots(ii) \end{cases}$

Here $f(x)$ is defined for $x \geq 1$ *i.e.,* on $[1, \infty)$ and also for $x < 1$ *i.e.,* on $(-\infty, 1)$.

Domain of f is $(-\infty, 1) \cup [1, \infty) = (-\infty, \infty) = R$

By (i), for all $x > 1$, $f(x) = x + 1$ is a polynomial and hence continuous.

By (ii), for all $x < 1$, $f(x) = x^2 + 1$ is a polynomial and hence continuous. Therefore f is continuous on $R - \{1\}$.

Let us examine continuity of f at the partitioning point

$\boldsymbol{x = 1.}$

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1)$ [By (ii)]

$(\because \; x \to 1^- \Rightarrow x < 1)$

Putting $x = 1$, $= 1^2 + 1 = 1 + 1 = 2$

Right Hand Limit = $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x + 1)$ [By (*i*)]

($\because$ $x \to 1^+ \Rightarrow x > 1$)

Putting $x = 1$, $= 1 + 1 = 2$

$\therefore \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$ (= 2)

$\therefore \lim\limits_{x \to 1} f(x)$ exists and = 2

Putting $x = 1$ in (*i*), $f(1) = 1 + 1 = 2$

$\therefore \lim\limits_{x \to 1} f(x) = f(1)$ (= 2)

$\therefore$ $f(x)$ is continuous at $x = 1$ also.
$\therefore$ f is be continuous on its whole domain (R here).
Hence no point of discontinuity.

11. $f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 \\ ^2 + 1, & \text{if } x > 2 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 & \ldots(i) \\ x^2 + 1, & \text{if } x > 2 & \ldots(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 2$ *i.e.,* on $(-\infty, 2]$ and also for $x > 2$ *i.e.,* on $(2, \infty)$.

$\therefore$ Domain of f is $(-\infty, 2] \cup (2, \infty) = (-\infty, \infty) = R$

By (*i*), for all $x < 2$, $f(x) = x^3 - 3$ is a polynomial and hence continuous.

By (*ii*), for all $x > 2$, $f(x) = x^2 + 1$ is a polynomial and hence continuous.

$\therefore$ f is continuous on R $- \{2\}$.

Let us examine continuity of f at the partitioning point $x = 2$.

Left Hand Limit = $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^3 - 3)$ [By (*i*)]

($\because$ $x \to 2^- \Rightarrow x < 2$)

Putting $x = 2$, $= 2^3 - 3 = 8 - 3 = 5$

Right Hand Limit = $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x^2 + 1)$ [By (*ii*)]

($\because$ $x \to 2^+ \Rightarrow x > 2$)

Putting $x = 2$, $= 2^2 + 1 = 4 + 1 = 5$

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \ (= 5)$

$\therefore \lim_{x \to 2} f(x)$ exists and $= 5$

Putting $x = 2$ in (i), $f(2) = 2^3 - 3 = 8 - 3 = 5$

$\therefore \lim_{x \to 2} f(x) = f(2) \ (= 5)$

$\therefore$ $f(x)$ is continuous at $x = 2$ (also).
Hence no point of discontinuity.

12. $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 & \ldots(i) \\ x^2, & \text{if } x > 1 & \ldots(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 1$ *i.e.,* on $(-\infty, 1]$ and also for $x > 1$ *i.e.,* on $(1, \infty)$.

$\therefore$ Domain of f is $(-\infty, 1] \cup (1, \infty) = (-\infty, \infty) = R$

By (i), for all $x < 1$, $f(x) = x^{10} - 1$ is a polynomial and hence continuous.

By (ii), for all $x > 1$, $f(x) = x^2$ is a polynomial and hence continuous.

$\therefore$ $f(x)$ is continuous on $R - \{1\}$.

Let us examine continuity of f at the partitioning point $x = 1$.

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^{10} - 1)$ [By (i)]

$(\because\ x \to 1^- \Rightarrow x < 1)$

Putting $x = 1$, $= (1)^{10} - 1 = 1 - 1 = 0$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2$ [By (ii)]

Putting $x = 1$, $= 1^2 = 1$

$\therefore \lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$

$\therefore \lim_{x \to 1} f(x)$ does not exist.

Hence the point of discontinuity is $x = 1$ (only).

13. Is the function defined by

$$f(x) = \begin{cases} x + 5 & \text{if } x \le 1 \\ x - 5 & \text{if } x > 1 \end{cases}$$

a continuous function?

Sol. Given: $f(x) = \begin{cases} x + 5, & \text{if } x \le 1 & \ldots(i) \\ x - 5, & \text{if } x > 1 & \ldots(ii) \end{cases}$

Here $f(x)$ is defined for $x \le 1$ *i.e.,* on $(-\infty, 1]$ and also for

$x > 1$ *i.e.,* on $(1, \infty)$

$\therefore$ Domain of f is $(-\infty, 1] \cup (1, \infty] = (-\infty, \infty) = R$.

By (*i*), for all $x < 1$, $f(x) = x + 5$ is a polynomial and hence continuous.

By (*ii*), for all $x > 1$, $f(x) = x - 5$ is a polynomial and hence continuous.

$\therefore$ f is continuous on R − {1}.

Let us examine continuity at the partitioning point $x = 1$.

Left Hand Limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x + 5)$ [By (i)]

Putting $x = 1$, $= 1 + 5 = 6$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x - 5)$ [By (ii)]

Putting $x = 1$, $= 1 - 5 = -4$

$\therefore \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$

$\therefore \lim\limits_{x \to 1} f(x)$ does not exist.

Hence $f(x)$ is discontinuous at $x = 1$.

$\therefore$ $x = 1$ is the only point of discontinuity.

Discuss the continuity of the function, f, where f is defined by

14. $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 & ...(i) \\ 4, & \text{if } 1 < x < 3 & ...(ii) \\ 5, & \text{if } 3 \le x \le 10 & ...(iii) \end{cases}$

From (i), (ii) and (iii), we can see that $f(x)$ is defined in [0, 1] ∪ (1, 3) ∪ [3, 10] *i.e.*, $f(x)$ is defined in [0, 10].

$\therefore$ Domain of $f(x)$ is [0, 10].

From (i), for $0 \le x < 1$, $f(x) = 3$ is a constant function and hence is continuous for $0 \le x < 1$.

From (ii), for $1 < x < 3$, $f(x) = 4$ is a constant function and hence is continuous for $1 < x < 3$.

From (iii), for $3 < x \le 10$, $f(x) = 5$ is a constant function and hence is continuous for $3 < x \le 10$.

Therefore, $f(x)$ is continuous in the domain [0, 10] − {1, 3}.

Let us examine continuity of f at the partitioning point $x = 1$.

Left Hand Limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 3$ [By (i)]

($\because$ $x \to 1^- \Rightarrow x < 1$)

Putting $x = 1$; $= 3$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 4$ [By (ii)]

($\because$ $x \to 1^+ \Rightarrow x > 1$)

Putting $x = 1$, $= 4$

$$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

$\therefore \lim_{x \to 1} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 1$.

Let us examine continuity of f at the partitioning point $x = 3$.

Left Hand Limit $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 4$ [By (*ii*)]

$(\because x \to 3^- \Rightarrow x < 3)$

Putting $x = 3$, $= 4$

Right Hand Limit $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 5$ [By (*iii*)]

$(\because x \to 3^+ \Rightarrow x > 3)$

Putting $x = 3$; $= 5$

$\therefore \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore \lim_{x \to 3} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 3$ also.

$\therefore x = 1$ and $x = 3$ are the two points of discontinuity of the function f in its domain [0, 10].

15. $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$.

Sol. The domain of f is $\{x \in R : x < 0\} \cup \{x \in R : 0 \le x \le 1\} \cup \{x \in R : x > 1\} = R$

$x = 0$ and $x = 1$ are partitioning points for the domain of this function.

For all $x < 0$, $f(x) = 2x$ is a polynomial and hence continuous.

For $0 < x < 1$, $f(x) = 0$ is a constant function and hence continuous.

For all $x > 1$, $f(x) = 4x$ is a polynomial and hence continuous.

Let us discuss continuity at partitioning point $x = 0$.

At $x = 0$, $f(0) = 0$ $[\because f(x) = 0 \text{ if } 0 \le x \le 1]$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x \; [\because x \to 0^- \Rightarrow x < 0 \text{ and } f(x) = 2x \text{ for } x < 0]$

$= 2 \times 0 = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0 \; [\because x \to 0^+ \Rightarrow x > 0 \text{ and } f(x) = 0 \text{ if } 0 \le x \le 1]$

$\therefore \lim_{x \to 0}$

$= 0$

$f(x) = \lim_{x \to 0^+} f(x)$

$= 0$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$ and hence f is continuous at 0.

Let us discuss continuity at partitioning point $x = 1$.

At $x = 1$, $f(1) = 0$ $\quad [\because f(x) = 0 \text{ if } 0 \le x \le 1]$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 0$ $[x \to 1- \Rightarrow x < 1$ and $f(x) = 0$ if $0 \le x \le 1]$

$= 0$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x$ $[x \to 1+ \Rightarrow x > 1$ and $f(x) = 4x$ for $x > 1]$

$= 4 \times 1 = 4$

The left and right hand limits of f at $x = 1$ do not coincide *i.e.,* are not equal.

$\therefore$ $\lim_{x \to 1} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = 1$.

Thus f is continuous at every point in the domain except $x = 1$. Hence, f is not a continuous function and $x = 1$ is the only point of discontinuity.

16. $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$.

Sol. Given: $f(x) = \begin{cases} -2, & \text{if } x \le -1 & \text{...}(i) \\ 2x, & \text{if } -1 < x \le 1 & \text{...}(ii) \\ 2, & \text{if } x > 1 & \text{...}(iii) \end{cases}$

From (i), (ii) and (iii) we can see that $f(x)$ is defined for

$\{x : x \le -1\} \cup \{x : -1 < x \le 1\} \cup \{x : x > 1\}$

i.e., for $(-\infty, -1] \cup (-1, 1] \cup (1, \infty) = (-\infty, \infty) = R$

$\therefore$ Domain of $f(x)$ is R.

From (i), for $x < -1$, $f(x) = -2$ is a constant function and hence is continuous for $x < -1$.

From (ii), for $-1 < x < 1$, $f(x) = 2x$ is a polynomial function and hence is continuous for $-1 < x < 1$.

From (iii), for $x > 1$, $f(x) = 2$ is a constant function and hence is continuous for $x > 1$.

Therefore $f(x)$ is continuous in R − {− 1, 1}.

Let us examine continuity of f at the partitioning point $x = -1$.

Left Hand Limit $= \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2)$ [By (i)]

Putting $x = -1$, $= -2$

$(\because \; x \to -1^- \;\Rightarrow\; x < -1)$

Right Hand Limit $= \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 2x$ (By (ii)]

$(\because \; x \to -1^+ \;\Rightarrow\; x > -1)$

Putting $x = -1$, $= 2(-1) = -2$

$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \; (= -2) \; \therefore \lim_{x \to -1} f(x)$ exists and $= -2$.

Putting $x = -1$ in (i), $f(-1) = -2$

$\therefore \lim_{x \to -1} f(x) = f(-1)\ (= -2)$ $\therefore$ $f(x)$ is continuous at $x = -1$.

Let us examine continuity of f at the partitioning point $x = 1$

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1} (2x)$ [By (*ii*)]

($\because$ $x \to 1^- \Rightarrow x < 1$)

Putting $x = 1$, $= 2(1) = 2$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2$ [By (*iii*)]

($\because$ $x \to 1^+ \Rightarrow x > 1$)

Putting $x = 1$, $= 2$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)\ (= 2)$ $\therefore \lim_{x \to 1} f(x)$ exists and $= 2$.

Putting $x = 1$ in (*ii*), $f(1) = 2(1) = 2$

$\therefore \lim_{x \to 1} f(x) = f(1)\ (= 2)$ $\therefore$ $f(x)$ is continuous at $x = 1$ also.

Therefore f is continuous for all x in its domain R.

17. Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ \quad + 3, & x > 3 \end{cases}$$

is continuous at $x = 3$.

Sol. Given: $$f(x) = \begin{cases} ax + 1 & \text{if } x \le 3 \quad \ldots(i) \\ bx + 3 & \text{if } x > 3 \quad \ldots(ii) \end{cases}$$

and $f(x)$ is continuous at $x = 3$.

Left Hand Limit $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1)$ [By (*i*)]

($x \to 3^- \Rightarrow x < 3$)

Putting $x = 3$, $= 3a + 1$ ...(*iii*)

Right Hand Limit $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3)$ [By (*ii*)]

($\because$ $x \to 3^+ \Rightarrow x > 3$)

Putting $x = 3$, $= 3b + 3$ ...(*iv*)

Putting $x = 3$ in (*i*), $f(3) = 3a + 1$ ...(*v*)

Because $f(x)$ is continuous at $x = 3$ (given)

$\therefore \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

Putting values from (*iii*), (*iv*) and (*v*) we have

$3a + 1 = 3b + 3$ $(= 3a + 1)$

$\therefore$ $3a + 1 = 3b + 3$ [$\because$ First and third members are equal]

$\Rightarrow$ $3a = 3b + 2$

Dividing by 3, $a = b + \frac{2}{3}$.

18. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

Sol. Given: $$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \quad \ldots(i) \\ 4x + 1, & \text{if } x > 0 \quad \ldots(ii) \end{cases}$$

Given: $f(x)$ is continuous at $x = 0$. To find λ.

Left Hand Limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \lambda(x^2 - 2x)$ [By (i)]

($\because x \to 0^- \Rightarrow x < 0$)

Putting $x = 0$, $= \lambda(0 - 0) = 0$

Right Hand Limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 1)$ [By (ii)]

($\because x \to 0^+ \Rightarrow x > 0$)

Putting $x = 0$, $= 4(0) + 1 = 1$

$\therefore \lim_{x \to 0^-} f(x)\ (= 0) \ne \lim_{x \to 0^+} f(x)\ (= 1)$

$\therefore \lim_{x \to 0} f(x)$ does not exist whatever λ may be

($\because$ Neither left limit nor right limit involves λ)

$\therefore$ For no value of λ, f is continuous at $x = 0$.

To examine continuity of f at $x = 1$

Left Hand Limit $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4x + 1)$ [By (ii)]

($x \to 1^- \Rightarrow x$ is slightly < 1 say $x = 0.99 > 0$)

Put $x = 1$, $= 4 + 1 = 5$

Right Hand Limit $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x + 1)$ [By (ii)]

($x \to 1^+ \Rightarrow x$ is slightly > 1 say $x = 1.1 > 0$)

Put $x = 1$, $= 4 + 1 = 5$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)\ (= 5)$

$\therefore \lim_{x \to 1} f(x)$ exists and $= 5$

Putting $x = 1$ in (ii) ($\because 1 > 0$), $f(1) = 4 + 1 = 5$)

$\therefore \lim_{x \to 1} f(x) = f(1)\ (= 5)$

$\therefore$ $f(x)$ is continuous at $x = 1$ (for all real values of λ).

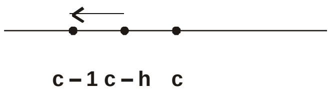
19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Sol. Given: $g(x) = x - [x]$

Let $x = c$ be any integer $(i.e., \ c \in Z\ (= I))$

Left Hand Limit $= \lim\limits_{x \to c^-} g(x) = \lim\limits_{x \to c^-} (x - [x])$

Put $x = c - h,\ h \to 0^+$



$$= \lim_{h \to 0^+} (c - h - [c - h])$$

$$= \lim_{h \to 0^+} (c - h - (c - 1))$$

$[\because$ If $c \in Z$ and $h \to 0^+$, then $[c - h] = c - 1]$

$$= \lim_{h \to 0^+} (c - h - c + 1) = \lim_{h \to 0^+} (1 - h)$$

Put $h = 0$, $= 1 - 0 = 1$

Right Hand Limit $= \lim_{x \to c^+} g(x) = \lim_{x \to c^+} (x - [x])$

Put $x = c + h$, $h \to 0^+$

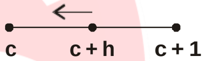
$$= \lim_{h \to 0^+} (c + h - [c + h]) = \lim_{h \to 0^+} (c + h - c)$$

$(\because$ If $c \in Z$ and $h \to 0^+$, then $[c + h] = c)$

$$= \lim_{h \to 0^+} h$$

Put $h = 0$; $= 0$

$\therefore \quad \lim_{x \to c^-} g(x) \neq \lim_{x \to c^+} g(x)$



$\therefore \quad \lim_{x \to c} g(x)$ does not exist and hence $g(x)$ is discontinuous at $x = c$ (any integer).

$\therefore \quad g(x) = x - [x]$ is discontinuous at all integral points.

Very Important Note. If two functions f and g are continuous in a common domain D,

then (*i*) $f + g$ (*ii*) $f - g$ (*iii*) fg are continuous in the same domain D.

(*iv*) $\dfrac{f}{g}$ is also continuous at all points of D except those where $g(x) = 0$.

20. Is the function $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$? Sol.

Given: $f(x) = x^2 - \sin x + 5 = (x^2 + 5) - \sin x$

$= g(x) - h(x)$ ...(*i*)

where $g(x) = x^2 + 5$ and $h(x) = \sin x$

We know that $g(x) = x^2 + 5$ is a polynomial function and hence is continuous (for all real x)

Again $h(x) = \sin x$ being a sine function is continuous (for all real x)

$\therefore$ By (*i*) $f(x) = x^2 - \sin x + 5 = g(x) - h(x)$

being the difference of two continuous functions is also continuous for all real x (see Note above) and hence continuous at $x = \pi (\in R)$ also.

Or

Given: $f(x) = x^2 - \sin x + 5$ ...(*i*)

To examine continuity at $x = \pi$

$$\lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5) \quad \text{[By } (i)]$$

Putting $x = \pi$, $= \pi^2 - \sin \pi + 5$

$= \pi^2 + 5$

$[\because \sin \pi = \sin 180° = \sin (180° - 0°) = \sin 0° = 0]$

Again putting $x = \pi$ in (i), $f(\pi) = \pi^2 - \sin \pi + 5$

$= \pi^2 - 0 + 5 = \pi^2 + 5$

$\therefore \lim_{x \to \pi} f(x) = f(\pi)$

$\therefore$ $f(x)$ is continuous at $x = \pi$.

21. iscuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \cdot \cos x$.

Sol. We know that sin x is a continuous function for all real x

Also we know that cos x is a continuous function for all real x (see solution of Q. No. 22(i) below)

$\therefore$ By Note at the end of solution of Q. No. 19,

(i) their sum function $f(x) = \sin x + \cos x$ is also continuous for all real x.

(ii) their difference function $f(x) = \sin x - \cos x$ is also continuous for all real x.

(iii) their product function $f(x) = \sin x \cdot \cos x$ is also continuous for all real x.

Note. To find $\lim_{x \to c} f(x)$, we can also start with putting $x = c + h$ where $h \to 0$ (and not only $h \to 0^+$)

$\therefore \lim_{x \to c} f(x) = \lim_{h \to 0} f(c + h)$.

(Please note that this method of finding the limits makes us find both $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ simultaneously).

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Sol. (i) Let $f(x)$ be the cosine function

i.e., $f(x) = \cos x$ ...(i)

Clearly, $f(x)$ is real and finite for all real values of x *i.e.*, $f(x)$ is defined for all real x. Therefore domain of $f(x)$ is R.

Let $x = c \in$ R.

$\lim_{x \to c} f(x) = \lim_{x \to c} \cos x$

Put $x = c + h$ where $h \to 0$

$= \lim_{h \to 0} \cos (c + h) = \lim_{h \to 0} (\cos c \cos h - \sin c \sin h)$

Putting $h = 0$, $= \cos c \cos 0 - \sin c \sin 0$

$= \cos c\,(1) - \sin c\,(0)$

$= \cos c$

$\therefore \lim_{x \to c} f(x) = \cos c$

Putting $x = c$ in (i), $f(c) = \cos c$

$\therefore \quad \lim_{x \to c} f(x) = f(c)$ (= cos c)

$\therefore$ $f(x)$ is continuous at (every) $x = c \in$ R

$\therefore$ $f(x) = \cos x$ is continuous on R.

(ii) Let $f(x)$ be cosecant function

i.e., $f(x) = \operatorname{cosec} x = \dfrac{1}{\sin x}$

$f(x)$ is not finite *i.e.,* $\to \infty$

when $\sin x = 0$ *i.e.,* when $x = n\pi$, $n \in$ Z.

$\therefore$ Domain of $f(x) = \operatorname{cosec} x$ is D = R − {$x = n\pi$; $n \in$ Z}.

($\because$ $f(x)$ is real and finite $\forall$ $x \in$ D).

Now $f(x) = \operatorname{cosec} x = \dfrac{1}{\sin x} = \dfrac{g(x)}{h(x)}$ ...(i)

Now $g(x) = 1$ being constant function is continuous on domain D and $h(x) = \sin x$ is non-zero and continuous on Domain D.

Therefore by (i), $f(x) = \operatorname{cosec} x \left(= \dfrac{1}{\sin x} = \dfrac{g(x)}{h(x)}\right)$ is continuous

on domain D = R − {$x = n\pi$, $n \in$ Z}

(Also read **Note** at the end of solution of Q. No. 19).

(iii) Let $f(x)$ be the secant function

i.e., $f(x) = \sec x = \dfrac{1}{\cos x}$ $f(x)$ is not finite *i.e.,* $\to \infty$

When $\cos x = 0$ *i.e.,* when $x = (2n + 1)\dfrac{\pi}{2}$, $n \in$ Z.

$\therefore$ Domain of $f(x) = \sec x$ is

$$D = R - \left\{x = (2n + 1)\frac{\pi}{2};\ n \in Z\right\}$$

Now $f(x) = \sec x = \dfrac{1}{\cos x} = \dfrac{g(x)}{h(x)}$ ...(i)

Now $g(x) = 1$ being constant function is continuous on domain D and $h(x) = \cos x$ is non-zero and continuous on domain D.

Therefore by (i), $f(x) = \sec x \left(= \dfrac{1}{\cos x} = \dfrac{g(x)}{h(x)}\right)$ is continuous

on domain D = R − $\left\{x : x = (2n + 1)\dfrac{\pi}{2};\ n \in Z\right\}$.

(iv) Let $f(x)$ be the cotangent function *i.e.,* $f(x) = \cot x = \dfrac{\cos x}{}$.

$\sin x$

$f(x)$ is not finite *i.e.,* $\rightarrow \infty$

When $\sin x = 0$ *i.e.,* when $x = n\pi, n \in Z$.

$\therefore$ Domain of $f(x) = \cot x$ is

$D = R - \{x = n\pi;\ n \in Z\}$

Now $f(x) = \cot x = \dfrac{\cos x}{\sin x} = \dfrac{g(x)}{h(x)}$ ...(*i*)

Now $g(x) = \cos x$ being cosine function is continuous on D and is non-zero on D.

Therefore by (*i*), $f(x) = \cot x \left(= \dfrac{\cos x}{\sin x} = \dfrac{g(x)}{h(x)}\right)$ is continuous on domain $D = R - \{x : x = n\pi, n \in Z\}$.

23. Find all points of discontinuity of *f*, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \geq 0 \end{cases}.$$

Sol. The domain of $f = \{x \in R : x < 0\} \cup \{x \in R : x \geq 0\} = R$

$x = 0$ is the partitioning point of the domain of the given function.

For all $x < 0$, $f(x) = \dfrac{\sin x}{x}$ (given)

Since $\sin x$ and x are continuous for $x < 0$ (in fact, they are continuous for all x) and $x \neq 0$

$\therefore f$ is continuous when $x < 0$

For all $x > 0$, $f(x) = x + 1$ is a polynomial and hence continuous.

$\therefore f$ is continuous when $x > 0$.

Let us discuss the continuity of $f(x)$ at the partitioning point $x = 0$.

At $x = 0$, $f(0) = 0 + 1 = 1$ $[\because f(x) = x + 1 \text{ for } x \geq 0]$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{\sin x}{x}$$

$$\left[\because x \to 0^- \Rightarrow x < 0 \text{ and } f(x) = \frac{\sin x}{x} \text{ for } x < 0\right]$$

$= 1$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1)$$

$$\left[\because x \to 0^+ \Rightarrow x > 0 \text{ and } f(x) = x + 1 \text{ for } x > 0\right]$$

$= 0 + 1 = 1$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1$. $\lim_{x \to 0} f(x) = 1$

Thus $\lim_{x \to 0} f(x) = f(0)$ and hence f is continuous at $x = 1$.

Now f is continuous at every point in its domain and hence f is a continuous function.

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

Sol. For all $x \neq 0$, $f(x) = x^2 \sin \frac{1}{x}$ being the product function of two continuous functions x^2 (polynomial function) and $\sin \frac{1}{x}$ (a sine function) is continuous for all real $x \neq 0$.

Now let us examine continuity at $x = 0$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x}$$

Putting $x = 0$ $= 0 \times$ A finite quantity between -1 and $1 = 0$

$$\left[\because \sin \frac{1}{x} \ (= \sin\theta) \text{ always lies between } -1 \text{ and } 1\right]$$

Also $f(x) = 0$ at $x = 0$ *i.e.*, $f(0) = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$, therefore function f is continuous at $x = 0$ (also).

Hence $f(x)$ continuous on domain R of f.

25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}.$$

Sol. Given: $f(x) = \begin{cases} \sin x - \cos x & \text{if } x \neq 0 & \quad ...(i) \\ -1 & \text{if } x = 0 & \quad ...(ii) \end{cases}$

From (i), $f(x)$ is defined for $x \neq 0$ and from (ii) $f(x)$ is defined for $x = 0$.

$\therefore$ Domain of $f(x)$ is $\{x : x \neq 0\} \cup \{0\} = \text{R}$.

From (i), for $x \neq 0$, $f(x) = \sin x - \cos x$ being the difference of two continuous functions $\sin x$ and $\cos x$ is continuous for all $x \neq 0$.

Hence $f(x)$ is continuous on $\text{R} - \{0\}$.

Now let us examine continuity at $x = 0$.

$$\lim_{x \to 0} f(x) = \lim_{x \to} (\sin x - \cos x)$$

[By (i) as $x \to 0$ means $x \neq 0$]

Putting $x = 0$, $= \sin 0 - \cos 0 = 0 - 1 = -1$

From (ii) $f(x) = -1$ when $x = 0$

i.e., $f(0) = -1$

$\therefore \quad \lim_{x \to 0} f(x) = f(0)\ (= -1)$

$\therefore$ $f(x)$ is continuous at $x = 0$ (also).

Hence $f(x)$ is continuous on domain R of f.

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

26. $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ **at** $x = \dfrac{\pi}{2}$.

Sol. Left Hand Limit $= \lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{k\cos x}{\pi - 2x}$

Put $x = \dfrac{\pi}{2} - h$ where $h \to 0^+$

$$= \lim_{h \to 0^+} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0^+} \frac{k\sin h}{\pi - \pi + 2h}$$

$$= \lim_{h \to 0^+} \frac{k\sin h}{2h} = \frac{k}{2} \times \lim_{h \to 0^+} \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \qquad \ldots(i)$$

Right Hand Limit $= \lim\limits_{x \to \frac{\pi}{2}^+} f(x) = \lim\limits_{x \to \frac{\pi}{2}^+} \dfrac{k\cos x}{\pi - 2x}$

Put $x = \dfrac{\pi}{2} + h$ where $h \to 0^+$

$$= \lim_{h \to 0^+} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0^+} \frac{-k\sin h}{\pi - \pi - 2h} = \lim_{h \to 0^+} \frac{-k\sin h}{-2h}$$

$$= \frac{k}{2} \times \lim_{h \to 0^+} \frac{\sin h}{h} = \frac{k}{2} \times 1 = \frac{k}{2} \qquad \ldots(ii)$$

Also $f\left(\dfrac{\pi}{2}\right) = 3 \quad \ldots(iii) \quad \because$

$f(x) = 3$ when $x = \frac{\pi}{2}$ (given)

Because $f(x)$ is continuous at $x = \frac{\pi}{2}$ (given)

$$\therefore \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

Putting values from (*i*), (*ii*), and (*iii*), $\frac{k}{2} = 3$ or $k = 6$.

27. $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ **at $x = 2$.**

Sol. Given: $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 & \ldots(i) \\ 3, & \text{if } x > 2 & \ldots(ii) \end{cases}$

Given: $f(x)$ is continuous at $x = 2$.

Left Hand Limit $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} kx^2$ [By (i)]

$(\because \ x \to 2^- \Rightarrow x \text{ is } < 2)$

Put $x = 2$, $= k(2)^2 = 4k$

Right Hand Limit $= \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} 3$ [By (ii)]

$(\because \ x \to 2^+ \Rightarrow x > 2)$

Putting $x = 2$, $= 3$

Putting $x = 2$ in (i) $f(2) = k(2)^2 = 4k$.

Because $f(x)$ is continuous at $x = 2$ (given),

therefore $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$

Putting values, $4k = 3 = 3 \Rightarrow k = \dfrac{3}{4}$.

28. $f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ **at $x = \pi$.**

Sol. Given: $f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi & \ldots(i) \\ \cos x, & \text{if } x > \pi & \ldots(ii) \end{cases}$

Given: $f(x)$ is continuous at $x = \pi$.

Left Hand Limit $= \lim\limits_{x \to \pi^-} f(x) = \lim\limits_{x \to \pi^-} (kx + 1)$ [By (i)]

$(\because \ x \to \pi^- \Rightarrow x < \pi)$

Putting $x = \pi$, $= k\pi + 1$

Right Hand Limit $= \lim\limits_{x \to \pi^+} f(x) = \lim\limits_{x \to \pi^+} \cos x$ [By (ii)]

$(\because \ x \to \pi^+ \Rightarrow x > \pi)$

Putting $x = \pi$, $= \cos \pi = \cos 180° = \cos (180° - 0)$

$= -\cos 0 = -1$

Putting $x = \pi$ in (i), $f(\pi) = k\pi + 1$

But $f(x)$ is continuous at $x = \pi$ (given), therefore

lim

$f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$

Putting values $k\pi + 1 = -1 = k\pi + 1$

$\Rightarrow \quad k\pi + 1 = -1$ [$\because$ First and third members are same]

$\Rightarrow \quad k\pi = -2 \Rightarrow k = -\frac{2}{\pi}$.

29. $f(x) = \begin{cases} kx+1, & \text{if } x \le 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$ **at $x = 5$.**

Sol. Given: $f(x) = \begin{cases} kx+1 & \text{if } x \le 5 & \text{...}(i) \\ 3x-5 & \text{if } x > 5 & \text{...}(ii) \end{cases}$

Given: $f(x)$ is continuous at $x = 5$.

Left Hand Limit $= \lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (kx + 1)$ [By (*i*)]

Putting $x = 5$, $= k(5) + 1 = 5k + 1$

Right Hand Limit $= \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (3x - 5)$ [By (*ii*)]

Putting $x = 5$, $= 3(5) - 5 = 15 - 5 = 10$

Putting $x = 5$ in (*i*), $f(5) = 5k + 1$

But $f(x)$ is continuous at $x = 5$ (given)

$\therefore \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$

Putting values $5k + 1 = 10 = 5k + 1$

$\Rightarrow 5k + 1 = 10 \Rightarrow 5k = 9 \Rightarrow k = \frac{9}{5}$.

30. Find the values of *a* and *b* such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

is a continuous function.

Sol. Given:
$$f(x) = \begin{cases} 5 & \text{if } x \le 2 & \ldots(i) \\ ax + b & \text{if } 2 < x < 10 & \ldots(ii) \\ 21 & \text{if } x \ge 10 & \ldots(iii) \end{cases}$$

From (*i*), (*ii*) and (*iii*), $f(x)$ is defined for $\{x \le 2\} \cup \{2 < x < 10\} \cup \{x \ge 10\}$ *i.e.*, for $(-\infty, 2] \cup (2, 10) \cup [10, \infty)$ *i.e.*, for $(-\infty, \infty)$ *i.e.*, on R. $\therefore$ Domain of $f(x)$ is R.

Given: $f(x)$ is a continuous function (of course on its domain here R), therefore $f(x)$ is also continuous at partitioning points $x = 2$ and $x = 10$ of the domain.

Because $f(x)$ is continuous at partitioning point $x = 2$, therefore

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \quad \ldots(iv)$$

Now $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5$ [By (*i*)]

$(\because x \to 2^- \Rightarrow x < 2)$

Putting $x = 2$, $= 5$

Again $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b)$ [By (*ii*)]

$(\because \ x \to 2^+ \ \Rightarrow \ x > 2)$

Putting $x = 2$, $= 2a + b$

Putting $x = 2$ in (*i*), $f(2) = 5$.

Putting these values in eqn. (*iv*), we have

$$5 = 2a + b = 5 \quad \Rightarrow \quad 2a + b = 5 \qquad \text{...}(v)$$

Again because $f(x)$ is continuous at partitioning point $x = 10$, therefore $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$ ...(*vi*)

Now $\lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} (ax + b)$ [By (*ii*)]

$(x \to 10^- \Rightarrow x < 10)$

Putting $x = 10$, $= 10a + b$

Again $\lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} 21$ [By (*iii*)]

$(\because x \to 10^+ \Rightarrow x > 10)$

Putting $x = 10$; $= 21$

Putting $x = 10$ in Eqn. (*iii*), $f(10) = 21$

Putting these values in eqn. (*vi*), we have

$10a + b = 21 = 21$

$\Rightarrow$ $10a + b = 21$ ...(*vii*)

Let us solve eqns. (*v*) and (*vii*) for a and b.

Eqn. (*vii*) – eqn. (*v*) gives $8a = 16 \Rightarrow a = \frac{16}{8} = 2$

Putting $a = 2$ in (*v*), $4 + b = 5$ $\therefore$ $b = 1$.

$\therefore$ $a = 2$, $b = 1$.

Very Important Result: Composite function of two continuous functions is continuous.

We know by definition that $(fog)x = f(g(x))$

and $(gof)x = g(f(x))$

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Sol. Given: $f(x) = \cos(x^2)$ ...(*i*)

$f(x)$ has a real and finite value for all $x \in$ R.

$\therefore$ Domain of $f(x)$ is R.

Let us take $g(x) = \cos x$ and $h(x) = x^2$.

Now $g(x) = \cos x$ is a cosine function and hence is continuous.

Again $h(x) = x^2$ is a polynomial function and hence is continuous.

$\therefore$ $(goh)x = g(h(x)) = g(x^2)$ $[\because h(x) = x^2]$

$= \cos(x^2)$ (Changing x to x^2 in $g(x) = \cos x$)

$= f(x)$ (By (*i*)) being the composite function of two continuous functions is continuous for all x in domain R.

Or

Take $g(x) = x^2$ and $h(x) = \cos x$.

Then $(hog)x = h(g(x)) = h(x^2)$

$= \cos(x^2) = f(x)$.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Sol. $f(x) = |\cos x|$ ...(*i*)

$f(x)$ has a real and finite value for all $x \in$ R.

$\therefore$ Domain of $f(x)$ is R.

Let us take $g(x) = \cos x$ and $h(x) = |x|$

We know that $g(x)$ and $h(x)$ being cosine function and modulus function are continuous for all real x.

Now $(goh)x = g(h(x)) = g(|x|) = \cos|x|$ being the composite function of two continuous functions is continuous (but $\neq f(x)$)

Again $(hog)x = h(g(x)) = h(\cos x)$

$= |\cos x| = f(x)$ [By (i)]

[Changing x to $\cos x$ in $h(x) = |x|$, we have $h(\cos x) = |\cos x|$]

Therefore $f(x) = |\cos x|$ (= $(hog)x$) being the composite functionof two continuous functions is continuous.

33. Examine that $\sin|x|$ is a continuous function.

Sol. Let $f(x) = \sin x$ and $g(x) = |x|$

We know that $\sin x$ and $|x|$ are continuous functions.

$\therefore$ f and g are continuous.

Now $(fog)(x) = f\{g(x)\} = \sin\{g(x)\} = \sin|x|$

We know that composite function of two continuous functions is continuous.

$\therefore$ fog is continuous. Hence, $\sin|x|$ is continuous.

34. Find all points of discontinuity of f defined by

$$f(x) = |x| - |x+1|.$$

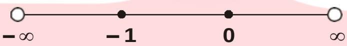
Sol. Given: $f(x) = |x| - |x+1|$ ...(i)

This $f(x)$ is real and finite for every $x \in R$.

$\therefore$ f is defined for all $x \in R$ *i.e.,* domain of f is R.

Putting each expression within modulus equal to 0

i.e., $x = 0$ and $x + 1 = 0$ *i.e.,* $x = 0$ and $x = -1$.



Marking these values of x namely -1 and 0 (in proper ascending order) on the number line, domain R of f is divided into three sub-intervals $(-\infty, -1]$, $[-1, 0]$ and $[0, \infty)$.

On the sub-interval $(-\infty, -1]$ *i.e.,* for $x \le -1$, (say for $x = -2$ etc.)

$x < 0$ and $(x + 1)$ is also < 0 and therefore

$|x| = -x$ and $|x+1| = -(x+1)$ Hence

(i) becomes $f(x) = |x| - |x+1|$

$= -x - (-(x+1)) = -x + x + 1$

i.e., $f(x) = 1$ for $x \le -1$ ...(ii)

On the sub-interval $[-1, 0]$ *i.e.,*for $-1 \le x \le 0$ $\left(\text{say for } x = \frac{-1}{2}\right)$

$x < 0$ and $(x + 1) > 0$ and therefore $|x| - x$ and $|x+1| = x + 1$.

Hence (i) becomes $f(x) = |x| - |x+1|$

$= -x - (x + 1) = -x - x - 1$

$$= -2x - 1$$

i.e., $f(x) = -2x - 1$ for $-1 \le x \le 0$ ...(*iii*)

On the sub-interval $[0, \infty)$ *i.e.,* for $x \ge 0$,

$x \ge 0$ and also $x + 1 > 0$ and therefore

$|x| = x$ and $|x + 1| = x + 1$ Hence

(*i*) becomes $f(x) = |x| - |x + 1| = x - (x + 1)$

$$= x - x - 1 = -1$$

i.e., $f(x) = -1$ for $x \ge 0$ ...(*iv*)

From (*ii*), for $x < -1$, $f(x) = 1$ is a constant function and hence is continuous for $x < -1$.

From (*iii*), for $-1 < x < 0$, $f(x) = -2x - 1$ is a polynomial function and hence is continuous for $-1 < x < 0$.

From (*iv*), for $x > 0$, $f(x) = -1$ is a constant function and hence is continuous for $x > 0$.

$\therefore$ f is continuous in $R - \{-1, 0\}$.

Let us examine continuity of *f* at partitioning point $x = -1$.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 1 \qquad \text{[By (ii)]}$$

$$(\because x \to -1^- \Rightarrow x < -1)$$

Putting $x = -1$, $= 1$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (-2x - 1) \qquad \text{(By (iii))}$$

$$(\because x \to -1^+ \Rightarrow x > -1)$$

Putting $x = -1$, $= -2(-1) - 1 = 2 - 1 = 1$

$$\therefore \quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \ (= 1)$$

$$\therefore \quad \lim_{x \to -1} f(x) \text{ exists and } = 1.$$

Putting $x = -1$ in (*ii*) or (*iii*), $f(-1) = 1$

$$\therefore \quad \lim_{x \to -1} f(x) = f(-1) \ (= 1)$$

$\therefore$ f is continuous at $x = -1$ also.

Let us examine continuity of *f* at partitioning point $x = 0$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-2x - 1) \qquad \text{(By (iii))}$$

$$(\because x \to 0^- \Rightarrow x < 0)$$

Putting $x = 0$, $= -2(0) - 1 = -1$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-1) \qquad \text{[By (iv)]}$$

$(\ldots \to 0^{+} \quad \Rightarrow \quad x > 0)$

Putting $x = 0$, $= -1$ $\therefore$ $\lim_{x \to 0^-} f(x) = \underset{x\to 0^+}{Lt} f(x)$ $(= -1)$

$\therefore$ $\lim_{x \to 0} f(x)$ exists and $= -1$

Putting $x = 0$ in (iii) or (iv), $f(0) = -1$

$\therefore$ $\lim_{x \to 0} f(x) = f(0)$ $(= -1)$

$\therefore$ f is continuous at $x = 0$ also.
$\therefore$ f is continuous on the domain R.
$\therefore$ There is no point of discontinuity.

Second Solution

We know that every modulus function is continuous for all real x.
Therefore $|x|$ and $|x + 1|$ are continuous for all real x.
Also, we know that difference of two continuous functions is continuous.

$\therefore$ $f(x) = |x| - |x + 1|$ is also continuous for all real x.
$\therefore$ There is no point of discontinuity.

Exercise 5.2

Differentiate the functions w.r.t. x in Exercises 1 to 8.

1. sin $(x^2 + 5)$.

Sol. Let $y = \sin(x^2 + 5)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^2+5) = \cos(x^2+5)\frac{d}{dx}(x^2+5)$$

$$\left[\because \frac{d}{dx}\sin f(x) = \cos f(x)\frac{d}{dx}f(x)\right]$$

$$= \cos(x^2+5)\,.\,(2x+0)$$

$$\left[\because \frac{d}{dx}x^n = n\,x^{n-1} \text{ and } \frac{d}{dx}(c) = 0\right]$$

$$= 2x\cos(x^2+5).$$

Caution. sin $(x^2 + 5)$ is not the product of two functions. It is composite function: sine of $(x^2 + 5)$.

2. cos (sin x).

Sol. Let $y = \cos(\sin x)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\cos(\sin x) = -\sin(\sin x)\frac{d}{dx}\sin x$$

$$\left[\because \frac{d}{dx}\cos f(x) = -\sin f(x)\frac{d}{dx}f(x)\right]$$

$$= -\sin(\sin x)\,.\,\cos x = -\cos x\sin(\sin x).$$

3. sin (*ax* + *b*).

Sol. Let $y = \sin(ax + b)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\sin(ax+b) = \cos(ax+b)\frac{d}{dx}(ax+b)$$

$$= \cos(ax+b)\left[a\frac{d}{dx}(x) + \frac{d}{dx}(b)\right]$$

$$= \cos(ax+b)\,[a(1)+0]$$

$$= a\cos(ax+b).$$

Note. It may be noted that letters a to q of English Alphabet are treated as constants (similar to 3, 5 etc.) as per convention.

4. sec (tan $\sqrt{x}$).

Sol. Let $y = \sec(\tan\sqrt{x})$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\sec(\tan\sqrt{x})$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\frac{d}{dx}(\tan\sqrt{x})$$

$$\left[\because \frac{d}{dx}\sec f(x) = \sec f(x)\tan f(x)\frac{d}{dx}f(x)\right]$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2(\sqrt{x})\frac{d}{dx}\sqrt{x}$$

$$\left[\because \frac{d}{dx}\tan f(x) = \sec^2 f(x)\frac{d}{dx}f(x)\right]$$

$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}\,\frac{1}{2\sqrt{x}}$$

$$\left[\because \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\right]$$

5. $\dfrac{\sin(ax+b)}{\cos(cx+d)}$.

Sol. Let $y = \dfrac{\sin(ax+b)}{\cos(cx+d)}$

$$\therefore \frac{dy}{dx} = \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{\cos^2(cx+d)}$$

$$\left[\because \text{By Quotient Rule } \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(\text{DEN.})\frac{d}{dx}(\text{NUM}) - \text{NUM}\frac{d}{dx}(\text{DEN})}{(\text{DEN})^2}\right]$$

$$= \frac{\cos(cx+d)\cos(ax+b)\frac{d}{dx}(ax+b) - \sin(ax+b)(-\sin(cx+d))\frac{d}{dx}(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(cx+d)\cos(ax+b) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$$\left[\because \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a\frac{d}{dx}(x) + 0 = a\,.\,1 = a \quad \text{Similarly } \frac{d}{dx}(cx+d) = c\right]$$

6. $\cos x^3 \sin^2 (x^5)$.

Sol. Let $y = \cos x^3 \sin^2 (x^5) = \cos x^3 (\sin x^5)^2$

$$\therefore \frac{dy}{dx} = \cos x^3 \frac{d}{dx}(\sin x^5)^2 + (\sin x^5)^2 \frac{d}{dx}\cos x^3$$

$$\left[\because \text{By Product Rule } \frac{d}{dx}(uv) = \text{I}\frac{d}{dx}(\text{II}) + \text{II}\frac{d}{dx}(\text{I})\right]$$

$$= \cos x^3 \,.\, 2(\sin x^5)\frac{d}{dx}\sin x^5 + (\sin x^5)^2(-\sin x^3)\frac{d}{dx}x^3$$

$$= \cos x^3 \,.\, 2(\sin x^5)\cos x^5 (5x^4) + \sin^2 x^5 (-\sin x^3)\, 3x^2$$

$$\left[\because \frac{d}{dx}\sin x^5 = \cos x^5 \frac{d}{dx}x^5 = \cos x^5 (5x^4)\right]$$

$$= 10x^4 \cos x^3 \sin x^5 \cos x^5 - 3x^2 \sin^2 x^5 \sin x^3$$

$$= x^2 \sin x^5 [10x^2 \cos x^3 \cos x^5 - 3\sin x^5 \sin x^3].$$

7. $2\sqrt{\cot(x^2)}$.

Sol. Let $y = 2\sqrt{\cot(x^2)} = 2(\cot(x^2))^{1/2}$

$$\therefore \frac{dy}{dx} = 2\,.\,\frac{1}{2}(\cot x^2)^{1/2-1}\frac{d}{dx}(\cot(x^2))$$

$$\left|\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right.$$

$$= (\cot x^2)^{-1/2}\left(-\operatorname{cosec}^2(x^2)\frac{d}{dx}x^2\right)$$

$$\left|\because \frac{d}{dx}\cot f(x) = -\operatorname{cosec}^2(f(x))\frac{d}{dx}f(x)\right.$$

$$= \frac{-\operatorname{cosec}^2(x^2)}{\sqrt{\cot x^2}}(2x) = \frac{-2x\operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}.$$

8. $\cos(\sqrt{x})$.

Sol. Let $y = \cos(\sqrt{x})$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\cos(\sqrt{x}) = -\sin\sqrt{x}\frac{d}{dx}\sqrt{x}$$

$$\left[\because \frac{d}{dx}\cos f(x) = -\sin f(x)\frac{d}{dx}f(x)\right]$$

$$= -\sin\sqrt{x}\ \frac{1}{2\sqrt{x}}\ \left[\because \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\right]$$

9. Prove that the function f given by $f(x) = |x - 1|$, $x \in \mathbb{R}$ is not differentiable at $x = 1$.

Sol. Definition. A function $f(x)$ is said to be differentiable at a point $x = c$ if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists

(and then this limit is called $f'(c)$ *i.e.,* value of $f'(x)$ or $\dfrac{dy}{dx}$ at $x = c$)

Here $f(x) = |x - 1|$, $x \in \mathbb{R}$ ...(*i*)

To prove: $f(x)$ is not differentiable at $x = 1$.

Putting $x = 1$ on (*i*), $f(1) = |1 - 1| = |0| = 0$

Left Hand Derivative $= Lf'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1}$

$$= \lim_{x \to 1^-} \frac{|x - 1| - 0}{x - 1} = \lim_{x \to 1^-} \frac{-(x - 1)}{x - 1}$$

$[\because\ x \to 1^- \Rightarrow x < 1 \Rightarrow x - 1 < 0 \Rightarrow |x - 1| = -(x - 1)]$

$$= \lim_{x \to 1^-} (-1) = -1 \qquad ...(ii)$$

Right Hand derivative $= Rf'(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1}$

$$= \lim_{x \to 1^+} \frac{|x - 1| - 0}{x - 1} = \lim_{x \to 1^+} \frac{(x - 1)}{x - 1}$$

$(\because\ x \to 1^+ \Rightarrow x > 1 \Rightarrow x - 1 > 0 \Rightarrow |x - 1| = x - 1)$

$$= \lim_{x \to 1^+} 1 = 1 \qquad ...(iii)$$

From (*ii*) and (*iii*), $Lf'(1) \neq Rf'(1)$

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

Note. In problems on limits of Modulus function, and bracket function (*i.e.,* greatest Integer Function), we have to find both left hand limit and right hand limit (we have used this concept quite few times in Exercise 5.1).

10. Prove that the greatest integer function defined by

$f(x) = [x]$, $0 < x < 3$

is not differentiable at $x = 1$ and $x = 2$.

Sol. Given: $f(x) = [x],\ 0 < x < 3$ ...(*i*)

Differentiability at $x = 1$

Putting $x = 1$ in (*i*), $f(1) = [1] = 1$

Left Hand derivative = $Lf'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{[x] - 1}{x - 1}$

Put $x = 1 - h,\ h \to 0^+$

$$= \lim_{h \to 0^+} \frac{[1-h]-1}{1-h-1} = \lim_{h \to 0^+} \frac{0-1}{-h} = \lim_{h \to 0^+} \frac{1}{h}$$

[We know that as $h \to 0^+$, $[c - h] = c - 1$ if c is an integer.
Therefore $[1 - h] = 1 - 1 = 0$]

Put $h = 0$, $= \frac{1}{0} = \infty$ does not exist.

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

(We need not find $Rf'(1)$ as $Lf'(1)$ does not exist).

Differentiability at $x = 2$

Putting $x = 2$ in (i), $f(2) = [2] = 2$

Left Hand derivative $= Lf'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{[x] - 2}{x - 2}$

Put $x = 2 - h$ as $h \to 0^+$

$$= \lim_{h \to 0^+} \frac{[2-h]-2}{2-h-2} = \lim_{h \to 0^+} \frac{1-2}{-h} = \lim_{h \to 0^+} \frac{-1}{-h}$$

(For $h \to 0^+$, $[2 - h] = 2 - 1 = 1$)

$$= \lim_{h \to 0^+} \frac{1}{h} = \frac{1}{0} = \infty \text{ does not exist.}$$

$\therefore$ $f(x)$ is not differentiable at $x = 2$.

Note. For $h \to 0^+$, $[c + h] = c$ if c is an integer.

Exercise 5.3

Find $\frac{dy}{dx}$ in the following Exercises 1 to 15.

1. $2x + 3y = \sin x$.

Sol. Given: $2x + 3y = \sin x$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin x$$

$$\therefore 2 + 3\frac{dy}{dx} = \cos x \Rightarrow 3\frac{dy}{dx} = \cos x - 2 \quad \therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}.$$

2. $2x + 3y = \sin y$.

Sol. Given: $2x + 3y = \sin y$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}\sin y \quad \therefore 2 + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$$

$$\Rightarrow -\cos y\frac{dy}{dx} + 3\frac{dy}{dx} = -2 \Rightarrow -\frac{dy}{dx}(\cos y - 3) = -2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}.$$

3. $ax + by^2 = \cos y$.

Sol. Given: $ax + by^2 = \cos y$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y) \quad \therefore a + b \cdot 2y\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow 2by\frac{dy}{dx} + \sin y\frac{dy}{dx} = -a$$

$$\Rightarrow \frac{dy}{dx}(2by + \sin y) = -a \quad \Rightarrow \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

4. $xy + y^2 = \tan x + y$.

Sol. Given: $xy + y^2 = \tan x + y$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}(xy) + \frac{d}{dx}y^2 = \frac{d}{dx}\tan x + \frac{d}{dx}y$$

Applying product rule,

$$x\frac{d}{dx}y + y\frac{d}{dx}x + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y \quad \therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}.$$

5. $x^2 + xy + y^2 = 100$.

Sol. Given: $x^2 + xy + y^2 = 100$

Differentiating both sides w.r.t. x,

$$\frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}y^2 = \frac{d}{dx}(100)$$

$$\therefore 2x + \left(x\frac{d}{dx}y + y\frac{d}{dx}x\right) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (x + 2y)\frac{dy}{dx} = -2x - y \quad \Rightarrow \frac{dy}{dx} = -\frac{(2x + y)}{x + 2y}.$$

6. $x^3 + x^2y + xy^2 + y^3 = 81$.

$dx \qquad x + 2y$

Sol. Given: $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides w.r.t. x,

$$\frac{d}{dx}x^3 + \frac{d}{dx}x^2y + \frac{d}{dx}xy^2 + \frac{d}{dx}y^3 = \frac{d}{dx}81$$

$$\therefore \; 3x^2 + \left(x^2\frac{dy}{dx} + y\cdot\frac{d}{dx}x^2\right) + x\frac{d}{dx}y^2 + y^2\frac{d}{dx}(x) + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + y \cdot 2x + x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2 + 2xy + 3y^2) = -3x^2 - 2xy - y^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}.$$

7. $\sin^2 y + \cos xy = \pi$.

Sol. Given: $\sin^2 y + \cos xy = \pi$

Differentiating both sides w.r.t. x,

$$\frac{d}{dx}(\sin y)^2 + \frac{d}{dx}\cos xy = \frac{d}{dx}(\pi)$$

$$\therefore \quad 2(\sin y)^1\frac{d}{dx}\sin y - \sin xy\frac{d}{dx}(xy) = 0$$

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$$

$$\Rightarrow \sin 2y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} - y\sin xy = 0$$

$$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}.$$

8. $\sin^2 x + \cos^2 y = 1$.

Sol. Given: $\sin^2 x + \cos^2 y = 1$

Differentiating both sides w.r.t. x,

$$\frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\cos y)^2 = \frac{d}{dx}(1)$$

$$\therefore \quad 2(\sin x)^1\frac{d}{dx}\sin x + 2(\cos y)^1\frac{d}{dx}\cos y = 0$$

$$\Rightarrow 2\sin x\cos x + 2\cos y\left(-\sin y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 2\sin x\cos x - 2\sin y\cos y\frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad -\sin 2y \frac{dy}{dx} = -\sin 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}.$$

9. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Sol. Given: $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

To simplify the given Inverse T-function, put $x = \tan\theta$.

$$\therefore \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\Rightarrow \quad y = 2\tan^{-1}x \qquad (\because \ x = \tan\theta \Rightarrow \theta = \tan^{-1}x)$$

$$\therefore \quad \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

10. $y = \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right), \ \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$.

Sol. Given: $y = \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right), \ \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$

To simplify the given Inverse T-function, put $x = \tan\theta$.

$$\therefore \quad y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta) = 3\theta$$

$$\Rightarrow \quad y = 3\tan^{-1}x \qquad (\because \ x = \tan\theta \Rightarrow \theta = \tan^{-1}x)$$

$$\therefore \quad \frac{dy}{dx} = 3 \cdot \frac{1}{1+x^2} = \frac{3}{1+x^2}.$$

11. $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), \ 0 < x < 1$.

Sol. Given: $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), \ 0 < x < 1$

To simplify the given Inverse T-function, put $x = \tan\theta$.

$$\therefore \quad y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$

$$= 2\theta = 2\tan^{-1}x \quad (\because \ x = \tan\theta \Rightarrow \theta = \tan^{-1}x)$$

$$\therefore \quad \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

12. $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), \ 0 < x < 1$.

Sol. Given: $y = \sin^{-1}\left(\dfrac{1-x^2}{}\right.$

$\left(1+x^2\right)$

To simplify the given Inverse T-function, **put $x = \tan\theta$.**

$$\therefore \quad y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1}\sin\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1}x \qquad (\because \quad x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}x)$$

$$\therefore \frac{dy}{dx} = 0 - 2 \cdot \frac{1}{1+x^2} = \frac{-2}{1+x^2}.$$

13. $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1.$

Sol. Given: $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

To simplify the given Inverse T-function **put $x = \tan\theta$.**

$$\therefore \quad y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\sin 2\theta)$$

$$= \cos^{-1}\cos\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1}x \quad (\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x)$$

$$\therefore \frac{dy}{dx} = 0 - 2 \cdot \frac{1}{1+x^2} = \frac{-2}{1+x^2}.$$

14. $y = \sin^{-1}(2x\sqrt{1-x^2}), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}.$

Sol. Given: $y = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta$

To simplify the given Inverse T-function,

put $x = \sin\theta$ (For $\sqrt{a^2 - x^2}$, put $x = a\sin\theta$)

$$\therefore \quad y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$= \sin^{-1}(2\sin\theta\sqrt{\cos^2\theta}) = \sin^{-1}(2\sin\theta\cos\theta)$$

$$y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$$

$$[\because x = \sin\theta \Rightarrow \theta = \sin^{-1}x]$$

$$\therefore \frac{dy}{dx} = 2 \cdot \frac{1}{\sqrt{1-x^2}}.$$

15. $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ $0 < x <$.

Sol. Given: $y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$

To simplify the given inverse T-function, **put $x = \cos\theta$.**

$$\therefore \quad y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1} x \quad (\because \; x = \cos\theta \;\Rightarrow\; \theta = \cos^{-1} x)$$

$$\therefore \quad \frac{dy}{dx} = 2\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-2}{\sqrt{1-x^2}}.$$

Exercise 5.4

Differentiate the following functions 1 to 10 w.r.t. x

1. $\dfrac{e^x}{\sin x}$.

Sol. Let $y = \dfrac{e^x}{\sin x}$

$$\therefore \frac{dy}{dx} = \frac{(\text{DEN})\frac{d}{dx}(\text{NUM}) - (\text{NUM})\frac{d}{dx}(\text{DEN})}{(\text{DEN})^2}$$

$$= \frac{\sin x \frac{d}{dx} e^x - e^x \frac{d}{dx} \sin x}{\sin^2 x} = \frac{\sin x \,.\, e^x - e^x \cos x}{\sin^2 x}$$

$$= e^x \frac{(\sin x - \cos x)}{\sin^2 x}.$$

2. $e^{\sin^{-1} x}$.

Sol. Let $y = e^{\sin^{-1} x}$

$$\therefore \frac{dy}{dx} = e^{\sin^{-1} x} \frac{d}{dx} \sin^{-1} x \qquad \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)\right]$$

$$= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}.$$

3. e^{x^3}.

Sol. Let $y = e^{x^3} = e^{(x^3)}$

$$\therefore \frac{dy}{dx} = e^{(x^3)} \frac{d}{dx} x^3 \qquad \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x)\right]$$

$$= e^{(x^3)}\, 3x^2 = 3x^2\, e^{(x^3)}.$$

4. $\sin(\tan^{-1} e^{-x})$.

Sol. Let $y = \sin(\tan^{-1} e^{-x})$

$$\therefore \frac{dy}{dx} = \cos(\tan^{-1} e^{-x}) \frac{d}{dx}$$

$$(\tan^{-1} e^{-x}) \quad \left[\because \frac{d}{dx} \sin f(x) = \cos f(x) \frac{d}{dx} f(x)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \frac{1}{1 + (e^{-x})^2} \frac{d}{dx} e^{-x}$$

$$\left[\because \frac{d}{dx} \tan^{-1} f(x) = \frac{1}{1 + (f(x))^2} \frac{d}{dx} f(x)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right) \frac{1}{1+e^{-2x}}\, e^{-x} \frac{d}{dx}(-x)$$

$$= -\frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \qquad \left[\because \frac{d}{dx}(-x) = -1\right]$$

5. $\log(\cos e^x)$.

Sol. Let $y = \log(\cos e^x)$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos e^x}\frac{d}{dx}(\cos e^x) \qquad \left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx} f(x)\right]$$

$$= \frac{1}{\cos e^x}(-\sin e^x)\frac{d}{dx}e^x \qquad \left[\because \frac{d}{dx}\cos f(x) = -\sin f(x)\frac{d}{dx} f(x)\right]$$

$$= -(\tan e^x)\, e^x = -e^x(\tan e^x)$$

6. $e^x + e^{x^2} + \ldots + e^{x^5}$.

Sol. Let $y = e^x + e^{x^2} + \ldots + e^{x^5}$

$$= e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}e^x + \frac{d}{dx}e^{x^2} + \frac{d}{dx}e^{x^3} + \frac{d}{dx}e^{x^4} + \frac{d}{dx}e^{x^5}$$

$$= e^x + e^{x^2}\frac{d}{dx}x^2 + e^{x^3}\frac{d}{dx}x^3 + e^{x^4}\frac{d}{dx}x^4$$

$$+ e^{x^5}\frac{d}{dx}x^5 \qquad \left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx} f(x)\right]$$

$$= e^x + e^{x^2} . 2x + e^{x^3} . 3x^2 + e^{x^4} . 4x^3 + e^{x^5}\, 5x^4$$

$$= e^x + 2x\, e^{x^2} + 3x^2\, e^{x^3} + 4x^3\, e^{x^4} + 5x^4\, e^{x^5}.$$

7. $\sqrt{e^{\sqrt{x}}}$, $x > 0$.

Sol. Let $y = \sqrt{e^{\sqrt{x}}} = \left(e^{\sqrt{x}}\right)^{1/2}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\left(e^{\sqrt{x}}\right)^{-1/2}\frac{d}{dx}e^{\sqrt{x}} \qquad \left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx} f(x)\right]$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}}\, e^{\sqrt{x}}\frac{d}{dx}\sqrt{x} \qquad \left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx} f(x)\right]$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} e^{\sqrt{x}} \frac{1}{2\sqrt{x}} \left[\because \frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \right]$$

$$= \frac{e^{\sqrt{x}}}{4\sqrt{x}\sqrt{e^{\sqrt{x}}}}.$$

8. log (log *x*), *x* > 1.

Sol. Let $y = \log(\log x)$

$$\therefore \frac{dy}{dx} = \frac{1}{\log x}\frac{d}{dx}(\log x) \qquad \left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{\log x}\cdot\frac{1}{x} = \frac{1}{x\log x}.$$

9. $\dfrac{\cos x}{\log x}, x > 0.$

Sol. Let $y = \dfrac{\cos x}{\log x}$

$$\therefore \frac{dy}{dx} = \frac{(\text{DEN})\frac{d}{dx}(\text{NUM}) - (\text{NUM})\frac{d}{dx}(\text{DEN})}{(\text{DEN})^2}$$

$$= \frac{\log x\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}\log x}{(\log x)^2}$$

$$= \frac{\log x(-\sin x) - \cos x\cdot\frac{1}{x}}{(\log x)^2}$$

$$= \frac{-\left(\sin x\log x + \frac{\cos x}{x}\right)}{(\log x)^2} = -\frac{(x\sin x\log x + \cos x)}{x(\log x)^2}.$$

10. $\cos(\log x + e^x), x > 0.$

Sol. Let $y = \cos(\log x + e^x)$

$$\therefore \frac{dy}{dx} = -\sin(\log x + e^x)\frac{d}{dx}(\log x + e^x)$$

$$\left[\because \frac{d}{dx}\cos f(x) = -\sin f(x)\frac{d}{dx}f(x)\right]$$

$$= -\sin(\log x + e^x)\cdot\left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x).$$

Exercise 5.5

Note. Logarithmic Differentiation.

The process of differentiating a function after taking its logarithm is called **logarithmic differentiation.**

This process of differentiation is very useful in the following situations:

(i) The given function is of the form $(f(x))^{g(x)}$

(ii) The given function involves complicated (as per our thinking) products (or and) quotients (or and) powers.

Remark 1. $\log \dfrac{(a^m\ b^n\ c^p)}{d^q\ l^k}$

$= m \log a + n \log b + p \log c - q \log d - k \log l$

Remark 2. $\log (u + v) \neq \log u + \log v$

and $\log (u - v) \neq \log u - \log v$.

Differentiate the following functions given in Exercises 1 to 5 w.r.t. *x*.

1. cos *x* cos 2*x* cos 3*x*.

Sol. Let $y = \cos x \cos 2x \cos 3x$ ...(*i*)

Taking logs on both sides, we have (see Note, (*ii*) page 261)

$$\log y = \log (\cos x \cos 2x \cos 3x)$$

$$= \log \cos x + \log \cos 2x + \log \cos 3x$$

Differentiating both sides w.r.t. *x*, we have

$$\frac{d}{dx} \log y = \frac{d}{dx} \log \cos x + \frac{d}{dx} \log \cos 2x + \frac{d}{dx} \log \cos 3x$$

$$\therefore \frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \frac{d}{dx} \cos x + \frac{1}{\cos 2x} \frac{d}{dx} \cos 2x$$

$$+ \frac{1}{\cos 3x} \frac{d}{dx} \cos 3x \quad \left[\because \frac{d}{dx} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x) \right]$$

$$= \frac{1}{\cos x} (-\sin x) + \frac{1}{\cos 2x} (-\sin 2x) \frac{d}{dx} (2x)$$

$$+ \frac{1}{\cos 3x} (-\sin 3x) \frac{d}{dx} 3x$$

$$= -\tan x - (\tan 2x)\, 2 - \tan 3x\, (3)$$

$$\therefore \frac{dy}{dx} = -y (\tan x + 2 \tan 2x + 3 \tan 3x).$$

Putting the value of y from (*i*),

$$\frac{dy}{dx} = -\cos x \cos 2x \cos 3x (\tan x + 2 \tan 2x + 3 \tan 3x).$$

2. $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$.

Sol. Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} = \left[\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]^{1/2}$ ...(*i*)

Taking logs on both sides, we have

$$\log y = \frac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)] \text{ (By Remark I above)}$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}\frac{d}{dx}(x-1) + \frac{1}{x-2}\frac{d}{dx}(x-2) - \frac{1}{x-3}\frac{d}{dx}(x-3) - \frac{1}{x-4}\frac{d}{dx}(x-4) - \frac{1}{x-5}\frac{d}{dx}(x-5)\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}y\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

Putting the value of y from (i),

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right].$$

3. $(\log x)^{\cos x}$**.**

Sol. Let $y = (\log x)^{\cos x}$ ...(i) [Form $(f(x))^{g(x)}$]

Taking logs on both sides of (i), we have (see Note (i) page 261)

$$\log y = \log (\log x)^{\cos x} = \cos x \log (\log x)$$

[$\because$ $\log m^n = n \log m$]

$$\therefore \quad \frac{d}{dx}\log y = \frac{d}{dx}[\cos x \,.\, \log (\log x)]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \cos x \frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\cos x$$

[By Product Rule]

$$= \cos x \frac{1}{\log x}\frac{d}{dx}\log x + \log(\log x)(-\sin x)$$

$$= \frac{\cos x}{\log x} \,.\, \frac{1}{x} - \sin x \log(\log x)$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{\cos x}{x\log x} - \sin x \log(\log x)\right].$$

Putting the value of y from (i),

$$\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right].$$

Very Important Note.

To differentiate $y = (f(x))^{g(x)} \pm (f(x))^{m(x)}$

or $\qquad y = (f(x))^{g(x)} \pm h(x)$

or $\qquad y = (f(x))^{g(x)} \pm k$ where k is a constant;

Never start with taking logs of both sides, put one term $= u$ and the other $= v$

$\therefore \qquad y = u \pm v$

$\therefore \qquad \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

Now find $\frac{du}{dx}$ and $\frac{dv}{dx}$ by the methods already learnt.

4. $x^x - 2^{\sin x}$.

Sol. Let $y = x^x - 2^{\sin x}$

Put $u = x^x$ and $v = 2^{\sin x}$ (See Note)

$\therefore$ $y = u - v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \quad \ldots(i)$$

Now $u = x^x$ [Form $(f(x))^{g(x)}$]

$\therefore$ $\log u = \log x^x = x \log x$ [$\because$ $\log m^n = n \log m$]

$$\therefore \quad \frac{d}{dx} \log u = \frac{d}{dx}(x \log x)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log x + \log x \frac{d}{dx} x$$

$$= x \cdot \frac{1}{x} + \log x \cdot 1 = 1 + \log x$$

$$\therefore \quad \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x) \quad \ldots(ii)$$

Again $v = 2^{\sin x}$

$$\therefore \quad \frac{dv}{dx} = \frac{d}{dx} 2^{\sin x} = 2^{\sin x} \log 2 \frac{d}{dx} \sin x$$

$$\left[\because \frac{d}{dx} a^{f(x)} = a^{f(x)} \log a \frac{d}{dx} f(x)\right]$$

$$\Rightarrow \quad \frac{dv}{dx} = 2^{\sin x}(\log 2) \cos x = \cos x \cdot 2^{\sin x} \log 2 \quad \ldots(iii)$$

Putting values from (*ii*) and (*iii*) in (*i*),

$$\frac{dy}{dx} = x^x(1 + \log x) - \cos x \cdot 2^{\sin x} \log 2.$$

5. $(x + 3)^2 (x + 4)^3 (x + 5)^4$.

Sol. Let $y = (x + 3)^2 (x + 4)^3 (x + 5)^4$ ...(*i*)

Taking logs on both sides of eqn. (*i*) (see Note (*ii*) page 261)

we have $\log y = 2 \log (x + 3) + 3 \log (x + 4)$
$+ 4 \log (x + 5)$ (By Remark I page 262)

$$\therefore \quad \frac{d}{dx} \log y = 2\frac{d}{dx}\log(x + 3) + 3\frac{d}{dx}\log(x + 4) + 4\frac{d}{dx}\log(x + 5)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 2 \cdot \frac{1}{x+3}\frac{d}{dx}(x + 3) + 3\frac{1}{x+4}\frac{d}{dx}(x+4)$$

$x + 4 \quad dx$

$$+ 4 \cdot \frac{1}{x+5} \frac{d}{dx}(x+5)$$

$$= \frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}$$

$$\therefore \quad \frac{dy}{dx} = y\left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right)$$

Putting the value of y from (i),

$$\frac{dy}{dx} = (x+3)^2 (x+4)^3 (x+5)^4 \left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right).$$

Differentiate the following functions given in Exercises 6 to 11 w.r.t. *x*.

6. $\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$.

Sol. Let $y = \left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Putting $\left(x + \frac{1}{x}\right)^x = u$ and $x^{\left(1+\frac{1}{x}\right)} = v$,

We have $y = u + v$ $\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(i)

Now $u = \left(x + \frac{1}{x}\right)^x$

Taking logarithms, $\log u = \log \left(x + \frac{1}{x}\right)^x = x \log \left(x + \frac{1}{x}\right)$ [Form uv]

Differentiating w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x+\frac{1}{x}} \frac{d}{dx}\left(x + \frac{1}{x}\right) + \log\left(x + \frac{1}{x}\right) . 1$$

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x+\frac{1}{x}} \cdot \left(1 - \frac{1}{x^2}\right) + \log\left(x + \frac{1}{x}\right) . 1$$

$$\left[\because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)\, x^{-2} = \frac{-1}{x^2}\right]$$

$$\Rightarrow \quad \frac{du}{dx} = u \left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right]$$

$$= \left(x + \frac{1}{x}\right)^x \left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right] \quad ...(ii)$$

Also $v = x^{\left(1+\frac{1}{x}\right)}$

Taking logarithms, $\log v = \log x^{\left(1+\frac{1}{x}\right)} = \left(1+\frac{1}{x}\right) \log x$

Differentiating w.r.t. x, we have

$$\frac{1}{v} \cdot \frac{dv}{dx} = \left(1+\frac{1}{x}\right) \cdot \frac{1}{x} + \log x \cdot \left(-\frac{1}{x^2}\right)$$

$$\left[\because \frac{d}{dx}\frac{1}{x} = \frac{d}{dx} x^{-1} = (-1)\, x^{-2} = \frac{-1}{x^2}\right]$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[\frac{1}{x}\left(1+\frac{1}{x}\right) - \frac{1}{x^2}\log x\right]$$

$$= x^{\left(1+\frac{1}{x}\right)}\left[\frac{1}{x}\left(1+\frac{1}{x}\right) - \frac{1}{x^2}\log x\right] \quad \ldots(iii)$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\frac{1}{x}\left(1+\frac{1}{x}\right) - \frac{1}{x^2}\log x\right]$$

7. $(\log x)^x + x^{\log x}$.

Sol. Let $y = (\log x)^x + x^{\log x}$

$= u + v$ where $u = (\log x)^x$ and $v = x^{\log x}$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots(i)$$

Now $u = (\log x)^x$ $\quad [(f(x))^{g(x)}]$

$\therefore \log u = \log (\log x)^x = x \log (\log x)$ $\quad [\because \log m^n = n \log m]$

$$\therefore \quad \frac{d}{dx}\log u = \frac{d}{dx}[x \log (\log x)]$$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}x \text{ (By product rule)}$$

$$= x \cdot \frac{1}{\log x}\frac{d}{dx}\log x + \log(\log x) \cdot 1$$

$$= x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x)$$

$$\therefore \quad \frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right] = (\log x)^x\left(\frac{1}{\log x} + \log(\log x)\right)$$

$$= (\log x)^x\frac{(1+\log x \log(\log x))}{\log x}$$

$$= (\log x)^{x-1}(1 + \log x \log(\log x)) \quad \ldots(ii)$$

Again $v = x^{\log x}$ $\quad$ [Form $(f(x))^{g(x)}$]

$\therefore \quad \log v = \log x^{\log x} = \log x \cdot \log x$ $\quad [\because \log m^n = n \log m]$

$= (\log x)^2$

$\therefore \frac{d}{dx} \log v = \frac{d}{dx} (\log x)^2$ $\quad \therefore \frac{1}{v} \frac{dv}{dx} = 2 (\log x)^1 \frac{d}{dx} \log x$

$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \frac{d}{dx} f(x)\right]$

$= 2 \log x . \frac{1}{x}$

$$\therefore \quad \frac{dv}{dx} = v\left(\frac{2}{x}\log x\right) = x^{\log x}\cdot\frac{2}{x}\log x$$

$$= 2x^{\log x - 1}\log x \qquad ...(iii)$$

Putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = (\log x)^{x-1}(1 + \log x \log(\log x)) + 2x^{\log x - 1}\log x.$$

8. $(\sin x)^x + \sin^{-1}\sqrt{x}$ **.**

Sol. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$= u + v$ where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now $u = (\sin x)^x$ [Form $(f(x))^{g(x)}$]

$\therefore \log u = \log(\sin x)^x = x \log \sin x$

$$\therefore \quad \frac{d}{dx}(\log u) = \frac{d}{dx}(x \log \sin x)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log \sin x + \log \sin x \frac{d}{dx}x$$

$$= x\cdot\frac{1}{\sin x}\frac{d}{dx}\sin x + (\log \sin x)\cdot 1$$

$$= x\frac{1}{\sin x}\cos x + \log \sin x = x \cot x + \log \sin x$$

$$\therefore \quad \frac{du}{dx} = u(x\cot x + \log \sin x) = (\sin x)^x(x\cot x + \log \sin x) \qquad ...(ii)$$

Again $v = \sin^{-1}\sqrt{x}$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}\sqrt{x} \quad \left|\because \frac{d}{dx}\sin^{-1}f(x) = \frac{1}{\sqrt{1-(f(x))^2}}\frac{d}{dx}f(x)\right.$$

$$= \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}} \quad \left[\because \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\right]$$

or $\frac{dv}{dx}$

$$= \frac{1}{2\sqrt{x - x^2}} = \frac{1}{2\sqrt{x}\,\sqrt{1-x}} = \frac{1}{2\sqrt{x(1-x)}} \quad \ldots(iii)$$

Putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*),

$$\frac{dy}{dx} = (\sin x)^x\,(x \cot x + \log \sin x) + \frac{1}{2\sqrt{x - x^2}}.$$

9. $\boldsymbol{x^{\sin x} + (\sin x)^{\cos x}}$.

Sol. Let $y = x^{\sin x} + (\sin x)^{\cos x}$

$= u + v$ where $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now $u = x^{\sin x}$ [Form $(f(x))^{g(x)}$]

$\therefore \log u = \log x^{\sin x} = \sin x \log x$

$$\therefore \frac{d}{dx} \log u = \frac{d}{dx} (\sin x \log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \sin x \frac{d}{dx} \log x + \log x \frac{d}{dx} \sin x$$

$$= \sin x \cdot \frac{1}{x} + (\log x) \cos x = \frac{\sin x}{x} + \cos x \log x$$

$$\therefore \frac{du}{dx} = u\left(\frac{\sin x}{x} + \cos x \log x\right)$$

$$= x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right) \qquad ...(ii)$$

Again $v = (\sin x)^{\cos x}$ [Form $f(x)^{g(x)}$]

$\therefore \log v = \log (\sin x)^{\cos x} = \cos x \log \sin x$

$$\therefore \frac{d}{dx}(\log v) = \frac{d}{dx}[\cos x \log \sin x]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \cos x \frac{d}{dx} \log \sin x + \log \sin x \frac{d}{dx} \cos x$$

$$= \cos x \frac{1}{\sin x} \frac{d}{dx}(\sin x) + \log \sin x (-\sin x)$$

$$= \cot x \cdot \cos x - \sin x \log \sin x$$

$$\therefore \frac{dv}{dx} = v (\cos x \cot x - \sin x \log \sin x)$$

$$= (\sin x)^{\cos x} (\cos x \cot x - \sin x \log \sin x) \qquad ...(iii)$$

Putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*),

we have $\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right)$

10. $x^{x \cos x} + \dfrac{x^2+1}{x^2-1}$.

$+ (\sin x)^{\cos x}$ $(\cos$ $x \cot x - \sin x \log \sin x)$

Sol. Let $y = x^{x \cos x} + \frac{x^2+1}{x^2-1}$

Putting $x^{x \cos x} = u$ and $\frac{x^2+1}{x^2-1} = v$

We have $y = u + v$ $\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ...(*i*)

Now $u = x^{x \cos x}$

Taking logarithms, $\log u = \log x^{x \cos x} = x \cos x \log x$

Differentiating w.r.t. x, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \cos x \log x)$$

$$= \frac{d}{dx}(x) \cdot \cos x \log x + x \frac{d}{dx}(\cos x) \cdot \log x + x \cos x \frac{d}{dx}(\log x)$$

$$\left[\because \frac{d}{dx}(uvw) = \frac{du}{dx} vw + u\frac{dv}{dx} \cdot w + uv\frac{dw}{dx}\right]$$

$$= 1 \cos x \log x + x(-\sin x)\log x + x \cos x \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{du}{dx} = u[\cos x \log x - x \sin x \log x + \cos x]$$

$$= x^{x \cos x}[\cos x \log x - x \sin x \log x + \cos x] \quad \ldots(ii)$$

Also $v = \dfrac{x^2+1}{x^2-1}$. Using quotient rule, we have

$$\frac{dv}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\cdot\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)\cdot 2x - (x^2+1)\cdot 2x}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}$$

$$= -\frac{4x}{(x^2-1)^2} \quad \ldots(iii)$$

Putting the values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = x^{x \cos x}[\cos x \log x - x \sin x \log x + \cos x] - \frac{4x}{(x^2-1)^2}.$$

11. $\mathbf{(x \cos x)^x + (x \sin x)^{1/x}}$**.**

Sol. Let $y = (x \cos x)^x + (x \sin x)^{1/x}$

Putting $(x \cos x)^x = u$ and $(x \sin x)^{1/x} = v$,

We have $y = u + v$ $\quad \therefore \quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} \quad \ldots(i)$

Now $u = (x \cos x)^x$

Taking logarithms, $\log u = \log (x \cos x)^x = x \log (x \cos x)$

$= x (\log x + \log \cos x)$

Differentiating w.r.t. x, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = x \left[\frac{1}{x} + \frac{1}{\cos x} \cdot (-\sin x) \right] + (\log x + \log \cos x) \cdot 1$$

$\Rightarrow \quad \frac{du}{dx} = u\,[1 - x\tan x + \log(x\cos x)]$

$[\because \log x + \log\cos x = \log(x\cos x)]$

$= (x\cos x)^x\,[1 - x\tan x + \log(x\cos x)]$ ...(ii)

Also $\quad v = (x\sin x)^{1/x}$

Taking logarithms, $\log v = \log(x\sin x)^{1/x} = \frac{1}{x}\log(x\sin x)$

$= \frac{1}{x}(\log x + \log\sin x)$

Differentiating w.r.t. x, we have

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x}\left[\frac{1}{x} + \frac{1}{\sin x}\cdot\cos x\right] + (\log x + \log\sin x)\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x\sin x)}{x^2}\right]$$

$$= (x\sin x)^{1/x}\cdot\left[\frac{1 + x\cot x - \log(x\sin x)}{x^2}\right] \quad \text{...(iii)}$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (ii) and (iii) in (i), we have

$$\frac{dy}{dx} = (x\cos x)^x\,[1 - x\tan x + \log(x\cos x)] + (x\sin x)^{1/x}\left[\frac{1 + x\cot x - \log(x\sin x)}{x^2}\right].$$

Find $\frac{dy}{dx}$ of the functions given in Exercises 12 to 15:

12. $x^y + y^x = 1$**.**

Sol. Given : $x^y + y^x = 1$

$\Rightarrow \quad u + v = 1$ where $u = x^y$ and $v = y^x$

$\therefore \quad \frac{d}{dx}(u) + \frac{d}{dx}(v) = \frac{d}{dx}(1)$

i.e., $\quad \frac{du}{dx} + \frac{dv}{dx} = 0$ ...(i)

Now $\quad u = x^y \quad [(\text{Variable})^{\text{variable}} = (f(x))^{g(x)}]$

$\therefore \quad \log u = \log x^y = y\log x$

$\therefore \quad \frac{d}{dx}\log u = \frac{d}{dx}$

$(y \log x)$ dx

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = y\frac{d}{dx}\log x + \log x \frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\therefore \frac{du}{dx} = u\left(\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right)$$

or $\quad \dfrac{du}{dx} = x^y\left(\dfrac{y}{x} + \log x\,\dfrac{dy}{dx}\right) = x^y\,\dfrac{y}{x} + x^y \log x\,\dfrac{dy}{dx}$

or $\quad \dfrac{du}{dx} = x^{y-1}y + x^y \log x\,\dfrac{dy}{dx}$ ...(*ii*) $\quad \left[\because \dfrac{x^y}{x} = \dfrac{x^y}{x^1} = x^{y-1}\right]$

Again $\quad v = y^x$

$\therefore \quad \log v = \log y^x = x \log y \qquad \therefore \dfrac{d}{dx}\log v = \dfrac{d}{dx}(x \log y)$

$\Rightarrow \quad \dfrac{1}{v}\dfrac{dv}{dx} = x\dfrac{d}{dx}(\log y) + \log y\,\dfrac{d}{dx}x = x\,.\,\dfrac{1}{y}\dfrac{dy}{dx} + \log y\,.\,1$

$\Rightarrow \quad \dfrac{dv}{dx} = v\left(\dfrac{x}{y}\dfrac{dy}{dx} + \log y\right)$

$= y^x\left(\dfrac{x}{y}\dfrac{dy}{dx} + \log y\right) = y^x\,\dfrac{x}{y}\dfrac{dy}{dx} + y^x \log y$

$\Rightarrow \quad \dfrac{dv}{dx} = y^{x-1}x\,\dfrac{dy}{dx} + y^x \log y$ ...(*iii*)

Putting values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$x^{y-1}y + x^y \log x\,\dfrac{dy}{dx} + y^{x-1}x\,\dfrac{dy}{dx} + y^x \log y = 0$$

or $\quad \dfrac{dy}{dx}(x^y \log x + y^{x-1}x) = -\,x^{y-1}\,y - y^x \log y$

$\therefore \quad \dfrac{dy}{dx} = -\,\dfrac{(x^{y-1}\,y + y^x \log y)}{x^y \log x + y^{x-1}\,x}.$

13. ***$y^x = x^y$.***

Sol. **Given:** $y^x = x^y \;\Rightarrow\; x^y = y^x$.

| Form on both sides is $(f(x))^{g(x)}$

Taking logarithms, $\log x^y = \log y^x \;\Rightarrow y \log x = x \log y$

Differentiating w.r.t. x, we have

$y\,.\,\dfrac{1}{x} + \log x\,.\,\dfrac{dy}{dx} = x\,.\,\dfrac{1}{y}\,.\,\dfrac{dy}{dx} + \log y\,.\,1$

$\Rightarrow \left(\log x - \dfrac{x}{y}\right)\dfrac{dy}{dx} = \log y - \dfrac{y}{x}$

$y \quad dx \quad x$

$$\Rightarrow \frac{y \log x - x}{y} \cdot \frac{dy}{dx} = \frac{x \log y - y}{x} \quad \therefore \frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}.$$

14. $(\cos x)^y = (\cos y)^x$.

Sol. **Given:** $(\cos x)^y = (\cos y)^x$ [Form on both sides is $(f(x))^{g(x)}$]

$\therefore$ Taking logs on both sides, we have

$$\log (\cos x)^y = \log (\cos y)^x$$

$$\Rightarrow \quad y \log \cos x = x \log \cos y \quad [\because \log m^n = n \log m]$$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}(y \log \cos x) = \frac{d}{dx}(x \log \cos y)$$

Applying Product Rule on both sides,

$$\Rightarrow y\frac{d}{dx} \log \cos x + \log \cos x \frac{dy}{dx}$$
$$= x\frac{d}{dx} \log \cos y + \log \cos y \frac{d}{dx} x$$

$$\Rightarrow y \cdot \frac{1}{\cos x} \frac{d}{dx} \cos x + \log \cos x \frac{dy}{dx}$$
$$= x \cdot \frac{1}{\cos y} \frac{d}{dx} \cos y + \log \cos y$$

$$\Rightarrow y\frac{1}{\cos x}(-\sin x) + \log \cos x \frac{dy}{dx}$$
$$= x\frac{1}{\cos y}\left(-\sin y \frac{dy}{dx}\right) + \log \cos y$$

$$\Rightarrow -y \tan x + \log \cos x \cdot \frac{dy}{dx} = -x \tan y \frac{dy}{dx} + \log \cos y$$

$$\Rightarrow x \tan y \frac{dy}{dx} + \log \cos x \cdot \frac{dy}{dx} = y \tan x + \log \cos y$$

$$\Rightarrow \frac{dy}{dx}(x \tan y + \log \cos x) = y \tan x + \log \cos y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}.$$

15. $xy = e^{x-y}$.

Sol. Given: $xy = e^{x-y}$

Taking logs on both sides, we have

$$\log (xy) = \log e^{x-y}$$

$\Rightarrow \log x + \log y = (x - y) \log e$

$\Rightarrow \log x + \log y = x - y$ $\quad (\because \log e = 1)$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx} \log x + \frac{d}{dx} \log y = \frac{d}{dx} x - \frac{d}{dx} y$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{1}{x} \Rightarrow \frac{dy}{dx}\left(\frac{1}{y} + 1\right) = \frac{x-1}{x}$

$\Rightarrow \left(\frac{1+y}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$

Cross-multiplying $x(1 + y)\frac{dy}{dx} = y(x - 1)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(x-1)}{x(1+y)}.$$

16. Find the derivative of the function given by $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ and hence find $f'(1)$.

Sol. Given: $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ ...(*i*)

Taking logs on both sides, we have

$$\log f(x) = \log (1 + x) + \log (1 + x^2) + \log (1 + x^4) + \log (1 + x^8)$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{1}{1+x}\frac{d}{dx}(1 + x) + \frac{1}{1+x^2}\frac{d}{dx}(1 + x^2) + \frac{1}{1+x^4}\frac{d}{dx}(1 + x^4) + \frac{1}{1+x^8}\frac{d}{dx}(1 + x^8)$$

$$\Rightarrow \frac{1}{f(x)}f'(x) = \frac{1}{1+x}\cdot 1 + \frac{1}{1+x^2}\cdot 2x + \frac{1}{1+x^4}\cdot 4x^3 + \frac{1}{1+x^8}\,8x^7$$

$$\therefore \quad f'(x) = f(x)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Putting the value of $f(x)$ from (*i*),

$$f'(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Putting $x = 1$,

$$f'(1) = (1 + 1)(1 + 1)(1 + 1)(1 + 1)\left[\frac{1}{1+1} + \frac{2}{1+1} + \frac{4}{1+1} + \frac{8}{1+1}\right]$$

$$= 2.2.2.2\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right] = 16\left[\frac{15}{2}\right] = 8 \times 15 = 120.$$

17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:

(i) **by using product rule.**

(ii) by expanding the product to obtain a single polynomial.

(iii) by logarithmic differentiation.

Do they all give the same answer?

Sol. Given: Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$ ...(1)

(i) To find $\frac{dy}{dx}$ by using Product Rule

$$\frac{dy}{dx} = (x^2 - 5x + 8)\ \frac{d}{dx}(x^3 + 7x + 9)$$

$$+ (x^3 + 7x + 9)\frac{d}{dx}(x^2 - 5x + 8)$$

$$= (x^2 - 5x + 8)(3x^2 + 7) + (x^3 + 7x + 9)(2x - 5)$$
$$= 3x^4 + 7x^2 - 15x^3 - 35x + 24x^2 + 56$$
$$+ 2x^4 - 5x^3 + 14x^2 - 35x + 18x - 45$$
$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11 \quad ...(2)$$

(ii) To find $\frac{dy}{dx}$ by expanding the product to obtain a single polynomial.

From (i), $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x$$
$$+ 8x^3 + 56x + 72$$

or $y = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11 \quad ...(3)$$

(iii) To find $\frac{dy}{dx}$ by logarithmic differentiation

Taking logs on both sides of (i), we have

$$\log y = \log (x^2 - 5x + 8) + \log (x^3 + 7x + 9)$$

$$\therefore \frac{d}{dx}\log y = \frac{d}{dx}\log (x^2 - 5x + 8) + \frac{d}{dx}\log (x^3 + 7x + 9)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}\frac{d}{dx}(x^2 - 5x + 8)$$

$$+ \frac{1}{x^3 + 7x + 9} \cdot \frac{d}{dx}(x^3 + 7x + 9)$$

$$= \frac{1}{x^2 - 5x + 8}(2x - 5) + \frac{1}{x^3 + 7x + 9}(3x^2 + 7)$$

$$\therefore \frac{dy}{dx} = y\left[\frac{(2x - 5)}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$$

$$= y\left[\frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}\right]$$

$$= y\,\frac{[2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^4 - 15x^3 + 24x^2 + 7x^2 - 35x + 56]}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

or $$\frac{dy}{dx} = y\,\frac{(5x^4 - 20x^3 + 45x^2 - 52x + 11)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

Putting the value of y from (i),

$$\frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9)\,\frac{(5x^4 - 20x^3 + 45x^2 - 52x + 11)}{(x^2 - 5x + 8)(x^3 + 7x + 9)}$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11 \qquad ...(4)$$

From (2), (3) and (4), we can say that value of $\frac{dy}{dx}$ is same obtained by three different methods used in (*i*), (*ii*) and (*iii*).

18. If *u, v* and *w* are functions of *x*, then show that

$$\frac{d}{dx}(u \cdot v \cdot w) = \frac{du}{dx} v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Sol. Given: u, v and w are functions of x.

To prove: $\frac{d}{dx}(u \cdot v \cdot w) = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$ ...(*i*)

(i) To prove eqn. (*i*): By repeated application of product rule

L.H.S. $= \frac{d}{dx}(u \cdot v \cdot w)$

Let us treat the product uv as a single function

$$= \frac{d}{dx}[(uv)w] \quad = uv\frac{d}{dx}(w) + w\frac{d}{dx}(uv)$$

Again Applying Product Rule on $\frac{d}{dx}(uv)$

L.H.S. $= \frac{d}{dx}(uvw) = uv\frac{dw}{dx} + w\left[u\frac{d}{dx}v + v\frac{d}{dx}u\right]$

$$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Rearranging terms

or $\frac{d}{dx}(uvw) = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$

which proves eqn. (*i*)

(ii) To prove eqn. (*i*): By Logarithmic differentiation

Let $y = uvw$

Taking logs on both sides

$$\log y = \log (u \cdot v \cdot w) = \log u + \log v + \log w$$

$$\therefore \frac{d}{dx}\log y = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$$

$dx \qquad u \;\; dx \qquad v \;\; dx \qquad w \;\; dx$

Putting $y = uvw$, $\dfrac{d}{dx}(uvw) = uvw\left(\dfrac{1}{u}\dfrac{du}{dx} + \dfrac{1}{v}\dfrac{dv}{dx} + \dfrac{1}{w}\dfrac{dw}{dx}\right)$

$$= \frac{du}{dx} . v . w + u . \frac{dv}{dx} . w + u .v. \frac{dw}{dx}$$ which proves eqn. (*i*).

Remark. The result of eqn. (*i*) can be used as a formula for derivative of product of three functions.
It can be used as a formula for doing Q. No. 1 and Q. No. 5 of this Exercise 5.5.

Exercise 5.6

If x and y are connected parametrically by the equations given in Exercises 1 to 5, without eliminating the parameter, find $\frac{dy}{dx}$.

1. $x = 2at^2$, $y = at^4$.

Sol. Given: $x = 2at^2$ and $y = at^4$

Differentiating both eqns. w.r.t. t, we have

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2) \qquad \text{and} \qquad \frac{dy}{dt} = \frac{d}{dt}(at^4)$$

$$= 2a\frac{d}{dt}t^2 \qquad\qquad = a\frac{d}{dt}t^4 = a.4t^3$$

$$= 2a.2t = 4at \qquad\qquad = 4at^3$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2$.

2. $x = a \cos \theta$, $y = b \cos \theta$.

Sol. Given: $x = a \cos \theta$ and $y = b \cos \theta$

Differentiating both eqns. w.r.t. θ, we have

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a \cos \theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = \frac{d}{d\theta}(b \cos \theta)$$

$$= a\frac{d}{d\theta}\cos \theta \qquad\qquad = b\frac{d}{d\theta}\cos \theta$$

$$= -a \sin \theta \qquad\qquad = -b \sin \theta$$

We know that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$.

3. $x = \sin t$, $y = \cos 2t$.

Sol. Given: $x = \sin t$ and $y = \cos 2t$

Differentiating both eqns. w.r.t. t, we have

$$\frac{dx}{dt} = \cos t \qquad \text{and} \qquad \frac{dy}{dt} = -\sin 2t\frac{d}{dt}(2t)$$

$$= -(\sin 2t)\, 2 = -2 \sin 2t$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\cos t}$

$$= -2 \cdot \frac{2 \sin t \cos t}{\cos t} = -4 \sin t.$$

4. $x = 4t,\ y = \frac{4}{t}$.

Sol. Given: $x = 4t$ and $y = \frac{4}{t}$

$\therefore \frac{dx}{dt} = \frac{d}{dt}(4t)$ and $\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right)$

$= 4\frac{d}{dt}t$ $= \dfrac{t\frac{d}{dt}(4) - 4\frac{d}{dt}t}{t^2}$

$= 4(1) = 4$ $= \dfrac{t(0) - 4(1)}{t^2} = -\dfrac{4}{t^2}$

We know that $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\left(-\frac{4}{t^2}\right)}{4} = \dfrac{-1}{(t^2)}$.

5. $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$.

Sol. Given: $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

$\therefore \frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}\cos 2\theta$ and $\frac{dy}{d\theta} = \cos\theta - \frac{d}{d\theta}\sin 2\theta$

$= -\sin\theta - (-\sin 2\theta)\frac{d}{d\theta}2\theta$ $= \cos\theta - \cos 2\theta\frac{d}{d\theta}2\theta$

$= -\sin\theta + (\sin 2\theta)\,2$ $= \cos\theta - \cos 2\theta(2)$

$= 2\sin 2\theta - \sin\theta$ $= \cos\theta - 2\cos 2\theta.$

We know that $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$.

If x and y are connected parametrically by the equations given in Exercises 6 to 10, without eliminating the parameter, find $\dfrac{dy}{dx}$.

6. $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$.

Sol. $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

Differentiating both eqns. w.r.t. θ, we have

$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\theta - \sin\theta)$ and $\frac{dy}{d\theta} = a\frac{d}{d\theta}(1 + \cos\theta)$

$= a\left[\frac{d}{d\theta}\theta - \frac{d}{d\theta}\sin\theta\right]$ and $\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}\cos\theta\right]$

$\Rightarrow \frac{dx}{d\theta} = a(1 - \cos\theta)$ and $\frac{dy}{d\theta} = a(0 - \sin\theta) = -a\sin\theta$

We know that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)}$

$$= -\frac{\sin\theta}{1-\cos\theta} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}.$$

7. $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$.

Sol. We have $x = \dfrac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\therefore \quad \frac{dx}{dt} = \frac{\sqrt{\cos 2t}\,.\frac{d}{dt}(\sin^3 t) - \sin^3 t\,.\frac{d}{dt}(\sqrt{\cos 2t})}{(\sqrt{\cos 2t})^2}$$

[By Quotient Rule]

$$= \frac{\sqrt{\cos 2t}\,.\,3\sin^2 t\frac{d}{dt}\,.\,(\sin t) - \sin^3 t\,.\,\frac{1}{2}(\cos 2t)^{-1/2}\,.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\,.\,3\sin^2 t\cos t - \dfrac{\sin^3 t}{2\sqrt{\cos 2t}}\,.\,(-2\sin 2t)}{\cos 2t}$$

$$= \frac{3\sin^2 t\cos t\cos 2t + \sin^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$= \frac{3\sin^2 t\cos t\cos 2t + \sin^3 t\,.\,2\sin t\cos t}{(\cos 2t)^{3/2}}$$

$$= \frac{\sin^2 t\cos t\,(3\cos 2t + 2\sin^2 t)}{(\cos 2t)^{3/2}}$$

and
$$\frac{dy}{dt} = \frac{\sqrt{\cos 2t}\,.\frac{d}{dt}(\cos^3 t) - \cos^3 t\,.\frac{d}{dt}(\sqrt{\cos 2t})}{(\sqrt{\cos 2t})^2}$$

[By Quotient Rule]

$$= \frac{\sqrt{\cos 2t}\,.\,3\cos^2 t\,.\frac{d}{dt}(\cos t) - \cos^3 t\,.\,\frac{1}{2}(\cos 2t)^{-1/2}\,.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\,.\,3\cos^2 t\,(-\sin t) - \dfrac{\cos^3 t}{2\sqrt{\cos 2t}}\,(-2\sin 2t)}{\cos 2t}$$

$$= \frac{-3\cos^2 t\sin t\cos 2t + \cos^3 t\,.\,\sin 2t}{(\cos 2t)^{3/2}}$$

$$= \frac{-3\cos^2 t\sin t\cos 2t + \cos^3 t\,.\,2\sin t\cos t}{(\cos 2t)^{3/2}}$$

$$= \frac{\sin t\cos^2 t\,(2\cos^2 t - 3\cos 2t)}{(\cos 2t)^{3/2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sin t \cos^2 t\,(2\cos^2 t - 3\cos 2t)}{(\cos 2t)^{3/2}} \cdot \frac{(\cos 2t)^{3/2}}{\sin^2 t \cos t\,(3\cos 2t + 2\sin^2 t)}$$

$$= \frac{\cos t\,[2\cos^2 t - 3(2\cos^2 t - 1)]}{\sin t\,[3(1 - 2\sin^2 t) + 2\sin^2 t]} = \frac{\cos t\,(3 - 4\cos^2 t)}{\sin t\,(3 - 4\sin^2 t)}$$

$$= \frac{-(4\cos^3 t - 3\cos t)}{3\sin t - 4\sin^3 t} = \frac{-\cos 3t}{\sin 3t} = -\cot 3t$$

Hence $\frac{dy}{dx} = -\cot 3t$.

8. $\boldsymbol{x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t.}$

Sol. $x = a\left[\cos t + \log\left(\tan\frac{t}{2}\right)\right]$

$$\Rightarrow \frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \cdot \frac{1}{\cos^2\frac{t}{2}} \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\left[\frac{1}{\sin t} - \sin t\right]$$

$$= a\left(\frac{1 - \sin^2 t}{\sin t}\right) = \frac{a\cos^2 t}{\sin t}$$

$$y = a\sin t \Rightarrow \frac{dy}{dt} = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\left(\frac{a\cos^2 t}{\ldots}\right)} \quad \frac{\sin t\cos t}{}$$

$= \tan t.$

sin t

9. $x = a \sec \theta, y = b \tan \theta$.

Sol. $x = a \sec \theta$ and $y = b \tan \theta$

Differentiating both eqns. w.r.t. θ, we have

$\frac{dx}{d\theta} = a \sec \theta \tan \theta$ and $\frac{dy}{d\theta} = b \sec^2 \theta$

We know that $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$

$$= \frac{b\cdot\frac{1}{\cos\theta}}{a\cdot\frac{\sin\theta}{\cos\theta}} = \frac{b}{\cos\theta}\cdot\frac{\cos\theta}{a\sin\theta} = \frac{b}{a\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta.$$

10. $\boldsymbol{x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta).}$

Sol. We have $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$\therefore \quad \frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta\,.\,1) = a\theta\cos\theta$

and $\quad \frac{dy}{d\theta} = a[\cos\theta - (\theta(-\sin\theta) + \cos\theta\,.\,1)]$

$= a\,[\cos\theta + \theta\sin\theta - \cos\theta] = a\theta\sin\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta.$$

11. **If** $\boldsymbol{x = \sqrt{a^{\sin^{-1}t}},\ y = \sqrt{a^{\cos^{-1}t}}}$**, show that** $\boldsymbol{\frac{dy}{dx} = -\frac{y}{x}}$**.**

Sol. Given: $x = \sqrt{a^{\sin^{-1}t}} = (a^{\sin^{-1}t})^{1/2} = a^{1/2\sin^{-1}t}$ ...(*i*)

$$\therefore \quad \frac{dx}{dt} = a^{1/2\sin^{-1}t}\log a\,\frac{d}{dt}\left(\frac{1}{2}\sin^{-1}t\right)$$

$$\left[\because \frac{d}{dx}a^x = a^x\log a \text{ and } \frac{d}{dx}a^{f(x)} = a^{f(x)}\log a\,\frac{d}{dx}f(x)\right]$$

$\Rightarrow \quad \frac{dx}{dt} = a^{1/2\sin^{-1}t}\log a\,.\,\frac{1}{2}\,\frac{1}{\sqrt{1-t^2}}$ ...(*ii*)

Again **given:** $y = \sqrt{a^{\cos^{-1}t}} = (a^{\cos^{-1}t})^{1/2} = a^{1/2\cos^{-1}t}$ ...(*iii*)

$$\therefore \quad \frac{dy}{dt} = a^{1/2\cos^{-1}t}\log a\,\frac{d}{dt}\left(\frac{1}{2}\cos^{-1}t\right)$$

$a^{1/2\cos^{-1}t}$ $\frac{1}{2}\left(\frac{-1}{\sqrt{\ }}\right)$ a

$\log a \cdot 2 \qquad \left(1-t^2 \right)$ ...(iv)

We know that $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

Putting values from (iv) and (ii),

$$\frac{dy}{dx} = \frac{a^{1/2\cos^{-1}t}\log a \cdot \dfrac{1}{2}\left(\dfrac{-1}{\sqrt{1-t^2}}\right)}{a^{1/2\sin^{-1}t}\log a \cdot \dfrac{1}{2}\dfrac{1}{\sqrt{1-t^2}}} = -\frac{a^{1/2\cos^{-1}t}}{a^{1/2\sin^{-1}t}} = -\frac{y}{x}$$

(By (iii) and (i))

Exercise 5.7

Find the second order derivatives of the functions given in Exercises 1 to 5.

1. $x^2 + 3x + 2$.

Sol. Let $y = x^2 + 3x + 2$

$$\therefore \frac{dy}{dx} = 2x + 3.1 + 0 = 2x + 3$$

Again differentiating w.r.t. x, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2(1) + 0 = 2.$

2. x^{20}.

Sol. Let $y = x^{20}$

$$\therefore \frac{dy}{dx} = 20x^{19}$$

Again differentiating w.r.t. x, $\frac{d^2y}{dx^2} = 20.19x^{18} = 380x^{18}$.

3. $x \cos x$.

Sol. Let $y = x \cos x$

$$\therefore \frac{dy}{dx} = x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}x \qquad \text{[By Product Rule]}$$

$$= -x \sin x + \cos x$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(x \sin x) + \frac{d}{dx}\cos x$$

$$= -\left[x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}(x)\right] - \sin x$$

$$= -(x \cos x + \sin x) - \sin x = -x \cos x - \sin x - \sin x$$

$$= -x \cos x - 2 \sin x = -(x \cos x + 2 \sin x).$$

4. $\log x$.

Sol. Let $y = \log x$ $\quad \therefore \frac{dy}{dx} = \frac{1}{x}$

Again differentiating w.r.t. x, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1}$

$= (-1)\, x^{-2} = \dfrac{-1}{x^2}.$

5. $\boldsymbol{x^3 \log x}$.

Sol. Let $y = x^3 \log x$

$$\therefore \quad \frac{dy}{dx} = x^3 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^3 \quad \text{[By Product Rule]}$$

$$= x^3 \cdot \frac{1}{x} + (\log x)\, 3x^2$$

$$= x^2 + 3x^2 \log x$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}x^2 + 3\frac{d}{dx}(x^2 \log x)$$

$$= 2x + 3\left[x^2 \frac{d}{dx}\log x + \log x \frac{d}{dx}x^2\right]$$

$$= 2x + 3\left(x^2 \cdot \frac{1}{x} + (\log x)\, 2x\right)$$

$$= 2x + 3(x + 2x \log x)$$

$$= 2x + 3x + 6x \log x = 5x + 6x \log x$$

$$= x(5 + 6 \log x).$$

Find the second order derivatives of the functions given in exercises 6 to 10.

6. $e^x \sin 5x$.

Sol. Let $y = e^x \sin 5x$

$$\therefore \frac{dy}{dx} = e^x \frac{d}{dx}\sin 5x + \sin 5x \frac{d}{dx}e^x \quad \text{[By Product Rule]}$$

$$= e^x \cos 5x \frac{d}{dx}5x + \sin 5x \,.\, e^x = e^x \cos 5x \,.\, 5 + e^x \sin 5x$$

or $\frac{dy}{dx} = e^x (5 \cos 5x + \sin 5x)$

Again applying Product Rule of derivatives

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(5 \cos 5x + \sin 5x) + (5 \cos 5x + \sin 5x)\frac{d}{dx}e^x$$

$$= e^x (5(-\sin 5x) \,.\, 5 + (\cos 5x) \,.\, 5) + (5 \cos 5x + \sin 5x)\, e^x$$

$$= e^x (-25 \sin 5x + 5 \cos 5x + 5 \cos 5x + \sin 5x)$$

$$= e^x (10 \cos 5x - 24 \sin 5x)$$

$$= 2e^x (5 \cos 5x - 12 \sin 5x).$$

7. $e^{6x} \cos 3x$.

Sol. Let $y = e^{6x} \cos 3x$

$$\therefore \frac{dy}{dx} = e^{6x} \frac{d}{dx}\cos 3x + \cos 3x \frac{d}{dx}e^{6x}$$

$$= e^{6x} (-\sin 3x)\frac{d}{dx}(3x) + \cos 3x \,.\, e^{6x}\frac{d}{dx}6x$$

$$= -e^{6x} \sin 3x \,.\, 3 + \cos 3x \,.\, e^{6x} \,.\, 6$$

$$\Rightarrow \frac{dy}{dx} = e^{6x} (-3 \sin 3x + 6 \cos 3x)$$

Again applying Product Rule of derivatives,

$$\frac{d^2y}{dx^2} = e^{6x} \frac{d}{dx}(-3 \sin 3x + 6 \cos 3x)$$

$+ (-3 \sin 3x$
$+ 6 \cos 3x)$ -

$\underline{d}$ e^{6x} dx

$= e^{6x}\,[-3 \,.\, \cos 3x \,.\, 3 - 6 \sin 3x \,.\, 3]$
$\qquad + (-3 \sin 3x + 6 \cos 3x)\, e^{6x} \,.\, 6$
$= e^{6x}\,(-9 \cos 3x - 18 \sin 3x - 18 \sin 3x + 36 \cos 3x)$
$= e^{6x}\,(27 \cos 3x - 36 \sin 3x)$
$= 9e^{6x}\,(3 \cos 3x - 4 \sin 3x).$

8. $\tan^{-1} x$.

Sol. Let $y = \tan^{-1} x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2}$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}(1) - 1\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)0 - (2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}.$$

9. $\log (\log x)$.

Sol. Let $y = \log (\log x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\log x}\frac{d}{dx}\log x \qquad \left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{\log x}\,\frac{1}{x} = \frac{1}{x \log x}$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{(x\log x)\frac{d}{dx}(1) - 1\frac{d}{dx}(x\log x)}{(x\log x)^2}$$

$$= \frac{(x\log x)\,0 - \left[x\frac{d}{dx}\log x + \log x\frac{d}{dx}(x)\right]}{(x\log x)^2}$$

$$= -\frac{\left[x\,.\,\frac{1}{x} + \log x\,.\,1\right]}{(x\log x)^2} = -\frac{(1+\log x)}{(x\log x)^2}.$$

10. $\sin (\log x)$.

Sol. Let $y = \sin(\log x)$

$$\therefore \frac{dy}{dx} = \cos(\log x)\frac{d}{dx}(\log x) = \cos(\log x) \cdot \frac{1}{x}$$

$$= \frac{\cos(\log x)}{x}$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{x\frac{d}{dx}\cos(\log x) - \cos(\log x)\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x[-\sin(\log x)]\frac{d}{dx}\log x - \cos(\log x)}{x^2}$$

$$= \frac{-x\sin(\log x)\cdot\frac{1}{x} - \cos(\log x)}{x^2} \quad = \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}.$$

11. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

Sol. Given: $y = 5\cos x - 3\sin x$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = -5\sin x - 3\cos x$

Again differentiating w.r.t. x, $\frac{d^2y}{dx^2} = -5\cos x + 3\sin x$

$= -(5\cos x - 3\sin x) \quad -y$ (By (*i*))

or $\frac{d^2y}{dx^2} = -y \quad \therefore \quad \frac{d^2y}{dx^2} + y = 0.$

12. If $y = \cos^{-1} x$. Find $\frac{d^2y}{dx^2}$ in terms of y alone.

Sol. Given: $y = \cos^{-1} x \quad \Rightarrow \quad x = \cos y$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-\cos^2 y}}$ (By (*i*))

$= \frac{-1}{\sqrt{\sin^2 y}} = \frac{-1}{\sin y} = -\operatorname{cosec} y$

or $\frac{dy}{dx} = -\operatorname{cosec} y$ ...(*ii*)

Again differentiating both sides w.r.t. x,

$\frac{d^2y}{dx^2} = -\frac{d}{dx}(\operatorname{cosec} y) = -\left[-\operatorname{cosec} y \cot y \frac{dy}{dx}\right]$

$= \operatorname{cosec} y \cot y\,(-\operatorname{cosec} y)$ (By (*ii*))

$= -\operatorname{cosec}^2 y \cot y.$

13. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$.

Sol. Given: $y = 3\cos(\log x) + 4\sin(\log x)$ ...(*i*)

$\therefore \quad \frac{dy}{dx} = (y_1) = -3\sin(\log x)\frac{d}{dx}\log x + 4\cos(\log x)\frac{d}{dx}\log x$

or $\quad y_1 = -3\sin(\log x) \cdot \frac{1}{x} + 4\cos(\log x) \cdot \frac{1}{x}$

Multiplying both sides by L.C.M. $= x$,

$xy_1 = -3\sin(\log x) + 4\cos(\log x)$

Again differentiating both sides w.r.t. x,

$$\frac{d}{dx}(xy_1) = -3\cos(\log x)\frac{d}{dx}\log x - 4\sin(\log x)\frac{d}{dx}\log x$$

$$\Rightarrow \quad x\frac{d}{dx}y_1 + y_1\frac{d}{dx}x = -3\cos(\log x)\cdot\frac{1}{x} - 4\sin(\log x)\cdot\frac{1}{x}$$

(By Product Rule)

$$\Rightarrow \quad xy_2 + y_1 = -\frac{[3\cos(\log x) + 4\sin(\log x)]}{x}$$

Cross-multiplying

$$x(xy_2 + y_1) = -[3\cos(\log x) + 4\sin(\log x)]$$

$$\Rightarrow \quad x^2y_2 + xy_1 = -y \qquad \text{(By } (i))$$

$$\Rightarrow \quad x^2y_2 + xy_1 + y = 0.$$

14. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.

Sol. Given: $y = Ae^{mx} + Be^{nx}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = Ae^{mx}\frac{d}{dx}(mx) + Be^{nx}\frac{d}{dx}(nx) \quad \left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$$

or $$\frac{dy}{dx} = Am\,e^{mx} + Bn\,e^{nx} \qquad ...(ii)$$

$$\therefore \quad \frac{d^2y}{dx^2} = Am.e^{mx}.m + Bne^{nx}.n$$

$$= Am^2\,e^{mx} + Bn^2\,e^{nx} \qquad ...(iii)$$

Putting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (*i*), (*ii*) and (*iii*) in

$$\text{L.H.S.} = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)(Am\,e^{mx} + Bn\,e^{nx}) + mn(Ae^{mx} + Be^{nx})$

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2\,e^{mx} - Bmn\,e^{nx} - Amn\,e^{mx}$
$- Bn^2\,e^{nx} + Amn\,e^{mx} + Bmn\,e^{nx} = 0 = \text{R.H.S.}$

15. If $y = 500\,e^{7x} + 600\,e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

Sol. Given: $y = 500\,e^{7x} + 600\,e^{-7x}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = 500\,e^{7x}(7) + 600\,e^{-7x}(-7) = 500(7)\,e^{7x} - 600(7)\,e^{-7x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = 500(7)\ e^{7x}\ (7) - 600(7)e^{-7x}(-7)$$

$$= 500(49)\ e^{7x} + 600(49)\ e^{-7x}$$

or $\quad \frac{d^2y}{dx^2} = 49[500\ e^{7x} + 600\ e^{-7x}] = 49y \qquad$ (By (i))

or $\quad \frac{d^2y}{dx^2} = 49y.$

16. If $e^y (x + 1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Sol. Given: $e^y (x + 1) = 1$

$\Rightarrow \quad e^y = \frac{1}{x+1}$

Taking logs of both sides, $\log e^y = \log \frac{1}{x+1}$

or $\quad y \log e = \log 1 - \log (x + 1)$

or $\quad y = -\log (x + 1) \quad [\because \log e = 1$ and $\log 1 = 0]$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{x+1}\frac{d}{dx}(x+1) = \frac{-1}{x+1} = -(x+1)^{-1}$

$\therefore \quad \frac{d^2y}{dx^2} = -(-1)(x+1)^{-2}\frac{d}{dx}(x+1)$ $\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$

$= \frac{1}{(x+1)^2}$ $\left[\because \frac{d}{dx}(x+1) = 1 + 0 = 1\right]$

L.H.S. $= \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

R.H.S. $= \left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$

$\therefore$ L.H.S. = R.H.S. *i.e.,* $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

17. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$.

Sol. Given: $y = (\tan^{-1} x)^2$ ...(*i*)

$\therefore \quad y_1 = 2(\tan^{-1} x)\frac{d}{dx}\tan^{-1} x$ $\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$

$\Rightarrow \quad y_1 = 2(\tan^{-1} x)\frac{1}{1+x^2} \quad \Rightarrow \quad y_1 = \frac{2\tan^{-1} x}{1+x^2}$

Cross-multiplying, $(1 + x^2) y_1 = 2 \tan^{-1} x$

Again differentiating both sides w.r.t. x,

$$(1 + x^2)\frac{d}{dx}y_1 + y_1\frac{d}{dx}(1 + x^2) = 2 \cdot \frac{1}{1 + x^2}$$

$\Rightarrow$
$$(1 + x^2)\, y_2 + y_1 \cdot 2x = \frac{2}{1 + x^2}$$

Multiplying both sides by $(1 + x^2)$,

$$(x^2 + 1)^2 y_2 + 2xy_1 (1 + x^2) = 2.$$

Exercise 5.8

$\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$, $\sin x$, $\cos x$, e^x, e^{-x}, $\log x$ $(x > 0)$ are continuous and derivable for all real x.

Note 2: Sum, difference, product of two continuous (derivable) functions is continuous (derivable).

1. Verify Rolle's theorem for $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.

Sol. Given: $f(x) = x^2 + 2x - 8$; $x \in [-4, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function of x (of degree 2).

$\therefore$ $f(x)$ is continuous and derivable everywhere *i.e.,* on $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval $[-4, 2]$ and derivable in open interval $(-4, 2)$.

Putting $x = -4$ in (*i*), $f(-4) = 16 - 8 - 8 = 0$

Putting $x = 2$ in (*i*), $f(2) = 4 + 4 - 8 = 0$

$\therefore$ $f(-4) = f(2)$ $(= 0)$

$\therefore$ All three conditions of Rolle's Theorem are satisfied.

From (*i*), $f'(x) = 2x + 2$.

Putting $x = c$, $f'(c) = 2c + 2 = 0 \Rightarrow 2c = -2$

$\Rightarrow$ $c = -\frac{2}{2} = -1 \in$ open interval $(-4, 2)$.

$\therefore$ Conclusion of Rolle's theorem is true.

$\therefore$ Rolle's theorem is verified.

2. Examine if Rolle's theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's theorem from these examples?

(*i*) $f(x) = [x]$ for $x \in [5, 9]$ (*ii*) $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Sol. (*i*) **Given:** $f(x) = [x]$ for $x \in [5, 9]$ ...(*i*)

(of course $[x]$ denotes the greatest integer $\leq x$)

We know that bracket function $[x]$ is discontinuous at all the integers (See Ex. 15, page 155, NCERT, Part I). Hence $f(x) = [x]$ is discontinuous at all integers between 5 and 9 *i.e.,* discontinuous at $x = 6$, $x = 7$ and $x = 8$ and hence discontinuous in the closed interval $[5, 9]$ and hence not derivable in the open interval $(5, 9)$. ...(*ii*) ($\because$ discontinuity $\Rightarrow$ Non-derivability)

Again from (*i*), $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore$ $f(5) \neq f(9)$

$\therefore$ Conditions of Rolle's Theorem are not satisfied.

$\therefore$ Rolle's Theorem is not applicable to $f(x) = [x]$ in the closed interval $[5, 9]$.

But converse (conclusion) of Rolle's theorem is true for this function $f(x) = [x]$.

i.e., $f'(c) = 0$ for every real c belonging to open interval

(5, 9) other than integers. (*i.e.,* for every real $c \neq 6, 7, 8$) (even though conditions are not satisfied).
Let us prove it.

Left Hand derivative $= Lf'(c) = \lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$

$= \lim\limits_{x \to c^-} \dfrac{[x] - [c]}{x - c}$ (By (*i*))

Put $x = c - h, h \to 0^+$, $= \lim\limits_{h \to 0^+} \dfrac{[c-h] - [c]}{c - h - c}$

$= \lim\limits_{h \to 0^+} \dfrac{[c] - [c]}{-h}$

[$\because$ We know that for $c \in R - Z$, as $h \to 0^+$, $[c - h] = [c]$]

$= \lim\limits_{h \to 0^+} \dfrac{0}{-h} = \lim\limits_{h \to 0^+} 0$

($\because$ $h \to 0^+ \Rightarrow h > 0$ and hence $h \neq 0$)

$= 0$ ...(*iii*)

Right Hand derivative $= Rf'(c) = \lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c}$

$= \lim\limits_{x \to c^+} \dfrac{[x] - [c]}{x - c}$ (By (*i*))

Put $x = c + h, h \to 0^+$, $= \lim\limits_{h \to 0^+} \dfrac{[c+h] - [c]}{c + h - c} = \lim\limits_{h \to 0^+} \dfrac{[c] - [c]}{h}$

[$\because$ We know that for $c \in R - Z$, as $h \to 0^+$, $[c + h] = [c]$]

$= \lim\limits_{h \to 0^+} \dfrac{0}{h} = \lim\limits_{h \to 0^+} 0$

($\because$ $h \to 0^+ \Rightarrow h > 0$ and hence $h \neq 0$)

$= 0$ ...(*iv*)

From (*iii*) and (*iv*) $Lf'(c) = Rf'(c) = 0$

$\therefore$ $f'(c) = 0 \ \forall$ real $c \in$ open interval (5, 9) other than integers $c = 6, 7, 8$.

(ii) **Given:** $f(x) = [x]$ for $x \in [-2, 2]$.

Reproduce the solution of (i) part replacing closed interval [5, 9] by [– 2, 2] and integers 6, 7, 8 by – 1, 0 and 1 lying between – 2 and 2.

(iii) **Given:** $f(x) = x^2 - 1$ for $x \in [1, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function of x (of degree 2).

$\therefore$ $f(x)$ is continuous and derivable everywhere *i.e.,* on $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval [1, 2] and derivable in the open interval (1, 2).

Again from (*i*), $f(1) = 1 - 1 = 0$

and $f(2) = 2^2 - 1 = 4 - 1 = 3$

$\therefore$ $f(1) \neq f(2)$.

$\therefore$ Conditions of Rolle's Theorem are not satisfied.

$\therefore$ Rolle's theorem is not applicable to $f(x) = x^2 - 1$ in $[1, 2]$.

Let us examine if converse (*i.e.,* conclusion) is true for this function given by (*i*).

From (*i*), $f'(x) = 2x$

Put $x = c$, $f'(c) = 2c = 0 \Rightarrow c = 0$ does not belong to open interval $(1, 2)$.

$\therefore$ Converse (conclusion) of Rolle's Theorem is also not true for this function.

3. If $f : [-5, 5] \to R$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Sol. Given: $f : [-5, 5] \to R$ is a differentiable function *i.e.,* f is differentiable on its domain closed interval $[-5, 5]$ (and in particular inopen interval $(-5, 5)$ also) and hence is continuous also on closed interval $[-5, 5]$ ...(*i*)

To prove: $f(-5) \neq f(5)$.

If possible, let $f(-5) = f(5)$ ...(*ii*)

From (*i*) and (*ii*) all the three conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists at least one point c in the open interval $(-5, 5)$ such that $f'(c) = 0$.

i.e., $f'(x) = 0$ *i.e.,* $f'(x)$ vanishes (vanishes $\Rightarrow$ zero) for at least one value of x in the open interval $(-5, 5)$. But this is contrary to given that $f'(x)$ does not vanish anywhere.

$\therefore$ Our supposition in (*ii*) *i.e.,* $f(-5) = f(5)$ is wrong.

$\therefore$ $f(-5) \neq f(5)$.

4. Verify Mean Value Theorem if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$ where $a = 1$ and $b = 4$.

Sol. Given: $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$ where $a = 1$ and $b = 4$ *i.e.,* in the interval $[1, 4]$ ...(*i*)

Here $f(x)$ is a polynomial function of x and hence is continuous and derivable everywhere.

$\therefore$ $f(x)$ is continuous in the closed interval $[1, 4]$ and derivable in the open interval $(1, 4)$ also.

$\therefore$ Both conditions of L.M.V.T. are satisfied.

From (*i*), $f'(x) = 2x - 4$

Put $x = c$, $f'(c) = 2c - 4$

from (*i*) $f(a) = f(1) = 1 - 4 - 3 = -6$

and $f(b) = f(4) = 16 - 16 - 3 = -3$

Putting these values in $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, we have

$$2c - 4 = \frac{-3 - (-6)}{4 - 1} \Rightarrow 2c - 4 = \frac{-3 + 6}{3}$$

3

$\Rightarrow$ $2c - 4 = \dfrac{}{3} = 1$ $\Rightarrow$ $2c = 5$

$\Rightarrow$ $c = \dfrac{5}{2} \in$ open interval $(1, 4)$.

$\therefore$ L.M.V.T. is verified.

5. Verify Mean Value Theorem if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$ where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Sol. Given: $f(x) = x^3 - 5x^2 - 3x$ ...(*i*)

In the interval $[a, b]$ where $a = 1$ and $b = 3$ *i.e.,* in the interval $[1, 3]$.

Here $f(x)$ is a polynomial function of x (of degree 3). Therefore, $f(x)$ is continuous and derivable everywhere *i.e.,* on the real line $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval $[1, 3]$ and derivable in open interval $(1, 3)$.

$\therefore$ Both conditions of Mean Value Theorem are satisfied.

From (*i*), $f'(x) = 3x^2 - 10x - 3$

Put $x = c$, $f'(c) = 3c^2 - 10c - 3$ ...(*ii*)

From (*i*), $f(a) = f(1) = 1 - 5 - 3 = 1 - 8 = -7$

and $f(b) = f(3) = 3^3 - 5 \,.\, 3^2 - 3.3 = 27 - 45 - 9 = 27 - 54 = -27$

Putting these values in the conclusion of Mean Value Theorem *i.e.,*

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ we have}$$

$$3c^2 - 10c - 3 = \frac{-27 - (-7)}{3 - 1} = \frac{-27 + 7}{2} = -\frac{20}{2} = -10$$

$\Rightarrow$ $3c^2 - 10c - 3 + 10 = 0$ $\Rightarrow$ $3c^2 - 10c + 7 = 0$

$\Rightarrow$ $3c^2 - 3c - 7c + 7 = 0$ $\Rightarrow$ $3c(c - 1) - 7(c - 1) = 0$

$\Rightarrow$ $(c - 1)(3c - 7) = 0$

$\therefore$ Either $c - 1 = 0$ or $3c - 7 = 0$

i.e., $c = 1 \notin$ open interval $(1, 3)$ or $3c = 7$ *i.e.,* $c = \dfrac{7}{3}$

which belongs to open interval $(1, 3)$.

Hence mean value theorem is verified.

Now we are to find all $c \in (1, 3)$ for which $f'(c) = 0$.

$\therefore$ From (ii), $3c^2 - 10c - 3 = 0$

Solving for c, $c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{100 + 36}}{6}$

$= \frac{10 \pm \sqrt{136}}{6} = \frac{10 \pm \sqrt{4 \times 34}}{6} = \frac{10 \pm 2\sqrt{34}}{6} = 2\left(\frac{5 \pm \sqrt{34}}{6}\right) = \frac{5 \pm \sqrt{34}}{3}$

Taking positive sign, $c = \frac{5 + \sqrt{34}}{3} > 3$ and hence $\notin (1, 3)$

Taking negative sign, $c = \frac{5 - \sqrt{34}}{3}$ is negative and hence $\notin (1, 3)$.

6. Examine the applicability of Mean Value Theorem for all the three functions being given below:
(*i*) $f(x) = [x]$ for $x \in [5, 9]$ (*ii*) $f(x) = [x]$ for $x \in [-2, 2]$
(*iii*) $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Sol. (*i*) *Reproduce solution of Q. No. 2(i) upto eqn. (ii)*

$\therefore$ Both conditions of L.M.V.T. are not satisfied.

$\therefore$ L.M.V.T. is not applicable to $f(x) = [x]$ for $x \in [5, 9]$.

(ii) *Reproduce solution of Q. No. 2(i) upto eqn. (ii)* replacing [5,9] by [– 2, 2] and integers 6, 7, 8 by – 1, 0 and 1 lying between – 2 and 2.

$\therefore$ Both conditions of L.M.V.T. are not satisfied.

$\therefore$ L.M.V.T. is not applicable to $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) **Given:** $f(x) = x^2 - 1$ for $x \in [1, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function (of degree 2).

Therefore $f(x)$ is continuous and derivable everywhere *i.e.,* on the real line $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval [1, 2] and derivable in open interval (1, 2).

$\therefore$ Both conditions of Mean Value Theorem are satisfied.

From (*i*), $f'(x) = 2x$

Put $x = c$, $f'(c) = 2c$

From (*i*), $f(a) = f(1) = 1^2 - 1 = 1 - 1 = 0$

$f(b) = f(2) = 2^2 - 1 = 4 - 1 = 3$

Putting these values in the conclusion of Mean Value Theorem *i.e.,* in $f'(c) = \frac{f(b) - f(a)}{b - a}$, we have

$= \quad \Rightarrow \quad c \quad 3$

$2c \qquad \Rightarrow \quad 2c = 3$

$= \frac{3}{}$

$\underline{-\ 0}$

$\in (1, 2)$

$2 \quad 1 \qquad -$

$\therefore$ Mean Value Theorem is verified.

MISCELLANEOUS EXERCISE

1. $(3x^2 - 9x + 5)^9$.

Sol. Let $y = (3x^2 - 9x + 5)^9$

$$\therefore \frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 \frac{d}{dx}(3x^2 - 9x + 5)$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= 9(3x^2 - 9x + 5)^8 [3(2x) - 9.1 + 0]$$

$$= 9(3x^2 - 9x + 5)^8 (6x - 9) = 27(3x^2 - 9x + 5)^8 (2x - 3).$$

2. $\sin^3 x + \cos^6 x$.

Sol. Let $y = \sin^3 x + \cos^6 x = (\sin x)^3 + (\cos x)^6$

$$\therefore \frac{dy}{dx} = 3(\sin x)^2 \frac{d}{dx}\sin x + 6(\cos x)^5 \frac{d}{dx}\cos x$$

$$\left[\because \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)\right]$$

$$= 3\sin^2 x \cos x - 6\cos^5 x \sin x$$

$$= 3\sin x \cos x(\sin x - 2\cos^4 x).$$

3. $(5x)^{3\cos 2x}$.

Sol. Let $y = (5x)^{3\cos 2x}$ ...(*i*) [Form $(f(x))^{g(x)}$]

Taking logs of both sides of (*i*) we have

$$\log y = \log (5x)^{3\cos 2x} = 3\cos 2x \log (5x)$$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}(\log y) = 3\frac{d}{dx}(\cos 2x \log (5x))$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \frac{d}{dx}\log (5x) + \log (5x)\frac{d}{dx}\cos 2x\right]$$

$$= 3\left[\cos 2x \cdot \frac{1}{5x}\frac{d}{dx}5x + \log (5x)(-\sin 2x)\frac{d}{dx}2x\right]$$

or $$\frac{1}{y}\frac{dy}{dx} = 3\left[\cos 2x \cdot \frac{1}{5x} \cdot 5 - 2\sin 2x \log 5x\right]$$

Cross-multiplying, $$\frac{dy}{dx} = 3y\left(\frac{\cos 2x}{x} - 2\sin 2x \log 5x\right)$$

Putting the value of y from (*i*),

$$\frac{dy}{dx} = 3(5x)^{3\cos 2x}\left(\frac{\cos 2x}{x} - 2\sin 2x \log 5x\right)$$

dx x

4. $\sin^{-1}(x\sqrt{x}),\ 0 \le x \le 1.$

Sol. Let $y = \sin^{-1}(x\sqrt{x}) = \sin^{-1}(x^{3/2})$

$\left[\because x\sqrt{x} = x^1 \cdot x^{1/2} = x^{1+1/2} = x^{3/2}\right]$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(x^{3/2})^2}} \frac{d}{dx} x^{3/2} \qquad \left[\because \frac{d}{dx}\sin^{-1} f(x) = \frac{1}{\sqrt{1-(f(x))^2}} \frac{d}{dx} f(x)\right]$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{3}{2} x^{1/2} = \frac{3\sqrt{x}}{2\sqrt{1-x^3}} = \frac{3}{2}\sqrt{\frac{x}{1-x^3}}.$$

5. $\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$**, $-2 < x < 2$.**

Sol. Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

Applying Quotient Rule,

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\frac{d}{dx}\cos^{-1}\frac{x}{2} - \cos^{-1}\frac{x}{2}\,\frac{d}{dx}\sqrt{2x+7}}{(\sqrt{2x+7})^2}$$

$$= \frac{\sqrt{2x+7}\left(\dfrac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\dfrac{d}{dx}\dfrac{x}{2}\right) - \left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{1}{2}(2x+7)^{-1/2}\dfrac{d}{dx}(2x+7)}{2x+7}$$

$$\left[\because \frac{d}{dx}\cos^{-1} f(x) = \frac{-1}{\sqrt{1-(f(x))^2}}\frac{d}{dx} f(x) \text{ and } \frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx} f(x)\right]$$

or $$\frac{dy}{dx} = \frac{-\sqrt{2x+7}\cdot\dfrac{2}{\sqrt{4-x^2}}\dfrac{1}{2} - \dfrac{1}{2}\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}\,2}{2x+7}$$

$$\left[\because \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \frac{1}{\sqrt{1-\frac{x^2}{4}}} = \frac{1}{\sqrt{\frac{4-x^2}{4}}} = \frac{2}{\sqrt{4-x^2}}\right]$$

$$= \frac{-\left[\dfrac{\sqrt{2x+7}}{\sqrt{4-x^2}} + \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}\right]}{2x+7} = -\left[\frac{2x+7+\sqrt{4-x^2}\cos^{-1}\frac{x}{2}}{\sqrt{4-x^2}\sqrt{2x+7}\,(2x+7)}\right]$$

$$= -\left[\frac{2x + 7 + \sqrt{4 - x^2}\cos^{-1}\left(\frac{x}{2}\right)}{\sqrt{4 - x^2}\,(2x + 7)^{3/2}}\right].$$

Differentiate w.r.t. x, the following functions in Exercises 6 to 11.

6. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, $0 < x < \dfrac{\pi}{2}$.

Sol. Let $y = \cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$ ...(i), $0 < x < \dfrac{\pi}{2}$

Let us simplify the given inverse T-function

Now $\sqrt{1+\sin x} = \sqrt{\cos^2\dfrac{x}{2}+\sin^2\dfrac{x}{2}+2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}$

$= \sqrt{\left(\cos\dfrac{x}{2}+\sin\dfrac{x}{2}\right)^2} = \cos\dfrac{x}{2}+\sin\dfrac{x}{2}$ ...(ii)

Again $\sqrt{1-\sin x} = \sqrt{\cos^2\dfrac{x}{2}+\sin^2\dfrac{x}{2}-2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}$

$= \sqrt{\left(\cos\dfrac{x}{2}-\sin\dfrac{x}{2}\right)^2} = \cos\dfrac{x}{2}-\sin\dfrac{x}{2}$

...(iii)

(**Given:** $0 < x < \dfrac{\pi}{2}$. Dividing by 2, $0 < \dfrac{x}{2} < \dfrac{\pi}{4}$ and therefore $\cos\dfrac{x}{2} > \sin\dfrac{x}{2} \Rightarrow \cos\dfrac{x}{2}-\sin\dfrac{x}{2} > 0$)

Putting values from (ii) and (iii) in (i), we have

$$y = \cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right) = \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$$

$= \cot^{-1}\left(\cot\dfrac{x}{2}\right) = \dfrac{x}{2}$ $\quad\therefore\quad \dfrac{dy}{dx} = \dfrac{1}{2}(1) = \dfrac{1}{2}$.

7. $(\log x)^{\log x}$, $x > 1$.

Sol. Let $y = (\log x)^{\log x}$, $x > 1$ ...(i) [Form $(f(x))^{g(x)}$]

Taking logs of both sides of (i), we have

$\log y = \log(\log x)^{\log x} = \log x \log(\log x)$ [$\because$ $\log m^n = n\log m$]

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\log x \log(\log x))$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \log x \frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\log x$$

(By Product Rule)

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \log x \cdot \frac{1}{\log x}\frac{d}{dx}\log x + \log(\log x) \cdot \frac{1}{x}$$

$$\left[\because \frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{\log(\log x)}{x} = \frac{1+\log(\log x)}{x}$$

$$\therefore \quad \frac{dy}{dx} = y\left(\frac{1+\log(\log x)}{x}\right)$$

Putting the value of y from (i), $\frac{dy}{dx} = (\log x)^{\log x}\left(\frac{1+\log(\log x)}{x}\right)$.

8. $\cos(a\cos x + b\sin x)$ for some constants a and b.

Sol. Let $y = \cos(a\cos x + b\sin x)$ for some constants a and b.

$$\therefore \quad \frac{dy}{dx} = -\sin(a\cos x + b\sin x)\frac{d}{dx}(a\cos x + b\sin x)$$

$$\left[\because \frac{d}{dx}\cos f(x) = -\sin f(x)\frac{d}{dx}f(x)\right]$$

$$= -\sin(a\cos x + b\sin x)[-a\sin x + b\cos x]$$

$$= -(-a\sin x + b\cos x)\sin(a\cos x + b\sin x)$$

$$= (a\sin x - b\cos x)\sin(a\cos x + b\sin x).$$

9. $(\sin x - \cos x)^{\sin x - \cos x}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$.

Sol. Let $y = (\sin x - \cos x)^{\sin x - \cos x}$ ...(i) [Form $(f(x))^{g(x)}$]

Taking logs of both sides of (i), we have

$$\log y = \log(\sin x - \cos x)^{(\sin x - \cos x)}$$

$$= (\sin x - \cos x)\log(\sin x - \cos x) \qquad [\because \log m^n = n\log m]$$

Differentiating both sides w.r.t. x, we have

$$\frac{d}{dx}\log y = (\sin x - \cos x)\frac{d}{dx}\log(\sin x - \cos x)$$

$$+ \log(\sin x - \cos x) \cdot \frac{d}{dx}(\sin x - \cos x)$$

(By Applying Product Rule on R.H. Side)

$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\sin$ ... $\cos$ ...

$$\frac{1}{(\sin x - \cos x)} \frac{d}{dx}(\sin x - \cos x)$$

$$+ \log(\sin x - \cos x) \, . \, (\cos x + \sin x)$$

$$\left[\because \frac{d}{dx} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x)\right]$$

$$= (\cos x + \sin x) + (\cos x + \sin x) \log(\sin x - \cos x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

$$\Rightarrow \quad \frac{dy}{dx} = y(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

Putting the value of y from (i),

$$\frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

10. **$x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.**

Sol. Let $y = x^x + x^a + a^x + a^a$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}x^x + \frac{d}{dx}x^a + \frac{d}{dx}a^x + \frac{d}{dx}a^a$$

$$= \frac{d}{dx}x^x + a\,x^{a-1} + a^x \log a + 0 \qquad ...(i)$$

[$\because$ a^a is constant as $3^3 = 27$ is constant]

To find $\frac{d}{dx}(x^x)$: Let $u = x^x$ ...(ii) $(f(x))^{g(x)}$]

$\therefore$ Taking logs on both sides of eqn. (ii), we have

$$\log u = \log x^x = x \log x$$

$$\therefore \quad \frac{d}{dx}\log u = \frac{d}{dx}(x \log x)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log x) + \log x \frac{d}{dx}x \qquad \text{(Product Rule)}$$

$$= x \cdot \frac{1}{x} + \log x \cdot 1 = 1 + \log x$$

$$\Rightarrow \quad \frac{du}{dx} = u(1 + \log x)$$

Putting the value of u from (ii), $\frac{d}{dx}x^x = x^x(1 + \log x)$

Putting this value in eqn. (i),

$$\frac{dy}{dx} = x^x(1 + \log x) + a\,x^{a-1} + a^x \log a.$$

11. **$x^{x^2-3} + (x-3)^{x^2}$ for $x > 3$.**

Sol. Let $y = x^{x^2-3} + (x-3)^{x^2}$ for $x > 3$

(**Caution.** For types $(f(x))^{g(x)} \pm (l(x))^{m(x)}$ or $(f(x))^{g(x)} \pm l(x)$

or $(f(x))^{g(x)} \pm k$ where k is a constant,

Never begin by taking logs of both sides as

$\log(m \pm n) \neq \log m \pm \log n$)

Put $u = x^{x^2 - 3}$ and $v = (x-3)^{x^2}$ $\therefore$ $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now $u = x^{(x^2 - 3)}$ [Type $(f(x))^{g(x)}$]

$\therefore$ Taking logs of both sides, we have

$$\log u = \log\ x^{(x^2-3)} = (x^2-3)\log x \qquad [\because\ \log m^n = n\log m]$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = (x^2-3)\frac{d}{dx}\log x + \log x\frac{d}{dx}(x^2-3)$$

$$= (x^2-3)\frac{1}{x} + \log x\,.\,(2x-0)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{x^2-3}{x} + 2x\log x \quad \therefore \frac{du}{dx} = u\left[\frac{x^2-3}{x} + 2x\log x\right]$$

Putting $u = x^{(x^2-3)}$, $\dfrac{du}{dx} = x^{(x^2-3)}\left(\dfrac{x^2-3}{x} + 2x\log x\right)$ ...(*ii*)

Again $v = (x-3)^{x^2}$ $[(f(x))^{g(x)}]$

$\therefore$ Taking logs of both sides, we have

$$\log v = \log\ (x-3)^{x^2} = x^2\log(x-3) \qquad [\because\ \log m^n = n\log m]$$

$$\therefore \frac{d}{dx}\log v = \frac{d}{dx}(x^2\log(x-3))$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = x^2\frac{d}{dx}\log(x-3) + \log(x-3)\frac{d}{dx}x^2$$

$$= x^2\frac{1}{x-3}\frac{d}{dx}(x-3) + \log(x-3)\,.\,2x$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{x^2}{x-3} + 2x\log(x-3)$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Putting $v = (x-3)^{x^2}$,

$$\frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right] \qquad ...(iii)$$

Putting values of $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ from (*ii*) and (*iii*) in (*i*), we have

$$\frac{dy}{dx} = x^{(x^2-3)}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right].$$

12. Find $\frac{dy}{dx}$ if $y = 12(1 - \cos t)$ and $x = 10(t - \sin t)$,

$$-\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Sol. Given: $y = 12(1 - \cos t)$ and $x = 10(t - \sin t)$

Differentiating both equations w.r.t. t, we have

$$\frac{dy}{dt} = 12\frac{d}{dt}(1-\cos t) \quad \text{and} \quad \frac{dx}{dt} = 10\frac{d}{dt}(t-\sin t)$$

$$= 12(0+\sin t) = 12\sin t \quad \text{and} \quad \frac{dx}{dt} = 10(1-\cos t)$$

We know that $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{12\sin t}{10(1-\cos t)}$

$$= \frac{6}{5}\cdot\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \frac{6}{5}\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}.$$

13. **Find $\dfrac{dy}{dx}$ if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$, $-1 \le x \le 1$.**

Sol. Given: $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}}\frac{d}{dx}\sqrt{1-x^2}$$

$$\left[\because \frac{d}{dx}\sin^{-1}f(x) = \frac{1}{\sqrt{1-(f(x))^2}}\frac{d}{dx}f(x)\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(1-x^2)}}\frac{1}{2}(1-x^2)^{-1/2}\frac{d}{dx}(1-x^2)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-1+x^2}}\frac{1}{2\sqrt{1-x^2}}(-2x)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x^2}}\left(\frac{-x}{\sqrt{1-x^2}}\right) = \frac{1}{\sqrt{1-x^2}} - \frac{x}{x\sqrt{1-x^2}}$$

$$\text{or} \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$$

14. **If $x\sqrt{1+y}$** $\sqrt{1+y}$ **Sol.** x

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$, prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$.

$x\sqrt{1+y} + y\sqrt{1+x} = 0.$ ...(*i*) (given)

We shall first find y in terms of x because y is not required in the value of $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$ to be proved.

From eqn. (*i*), $x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, $x^2(1+y) = y^2(1+x)$

or $x^2 + x^2y = y^2 + y^2x$ or $x^2 - y^2 = -x^2y + y^2x$

or $\quad (x - y)(x + y) = - xy(x - y)$

Dividing both sides by $\quad (x - y) \neq 0 \qquad (\because x \neq y)$

$$x + y = - xy \text{ or } y + xy = - x$$

$\Rightarrow \qquad y(1 + x) = - x \quad \therefore \quad y = -\dfrac{x}{1+x}$

Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= -\frac{(1+x).1 - x.1}{(1+x)^2} = -\frac{1}{(1+x)^2}.$$

15. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

Sol. The given equation is $(x - a)^2 + (y - b)^2 = c^2$ ...(*i*)

Differentiating both sides of eqn. (*i*) w.r.t. x,

$$2(x - a) + 2(y - b)\frac{dy}{dx} = 0$$

or $\quad 2(y - b)\dfrac{dy}{dx} = - 2(x - a) \quad \therefore \quad \dfrac{dy}{dx} = -\left(\dfrac{x-a}{y-b}\right)$ ...(*ii*)

Again differentiating both sides of (*ii*) w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{-\left[(y-b).1 - (x-a)\frac{dy}{dx}\right]}{(y-b)^2}$$

Putting the value of $\dfrac{dy}{dx}$ from (*i*),

$$\frac{d^2y}{dx^2} = \frac{-\left[(y-b) - (x-a)\left(\frac{-(x-a)}{y-b}\right)\right]}{(y-b)^2} = \frac{-\left[(y-b) + \frac{(x-a)^2}{y-b}\right]}{(y-b)^2}$$

$$= \frac{-[(y-b)^2 + (x-a)^2]}{(y-b)^3} \qquad dy$$

[By (i)]

$= \dfrac{-a^2}{(y-b)^3}$ ...(iii)

Putting values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (ii) and (iii) in the given expression

$$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}, \text{ it is } = -\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{3/2}}{\frac{c^2}{(y-b)^3}}$$

$$= \frac{[(y-b)^2+(x-a)^2]^{3/2}}{(y-b)^3} \times \frac{(y-b)^3}{-c^2} \qquad [\because ((y-b)^2)^{3/2} = (y-b)^3]$$

Putting $(x - a)^2 + (y - b)^2 = c^2$ from (i)

$$= \frac{(c^2)^{3/2}}{-c^2} = \frac{-c^3}{c^2} = -c$$

which is a constant and is independent of a and b.

16. If $\cos y = x \cos (a + y)$ with $\cos a \neq \pm 1$, prove that

$$\frac{dy}{dx} = \frac{\cos^2 (a+y)}{\sin a}.$$

Sol. Given: $\cos y = x \cos (a + y)$

$$\therefore \qquad x = \frac{\cos y}{\cos (a+y)} \qquad ...(i)$$

(We have found the value of x because x is not present in the required value of $\frac{dy}{dx}$)

Differentiating both sides of (i) w.r.t. y, $\frac{dx}{dy} = \frac{d}{dy}\left(\frac{\cos y}{\cos (a+y)}\right)$

Applying Quotient Rule,

$$\frac{dx}{dy} = \frac{\cos (a+y)\frac{d}{dy}\cos y - \cos y \frac{d}{dy}\cos (a+y)}{\cos^2 (a+y)}$$

or $$\frac{dx}{dy} = \frac{\cos (a+y)(-\sin y) - \cos y(-\sin (a+y))}{\cos^2 (a+y)}$$

$$\left[\because \frac{d}{dy}\cos (a+y) = -\sin (a+y)\frac{d}{dy}(a+y) = -\sin (a+y)(0+1) = -\sin (a+y)\right]$$

or $$\frac{dx}{dy} = \frac{-\cos (a+y)\sin y + \sin (a+y)\cos y}{\cos^2 (a+y)}$$

$$= \frac{\sin (a+y)\cos y - \cos (a+y)\sin y}{}$$

$\cos^2(a+y)$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

$[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$

Taking reciprocals $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

17. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

Sol. Given: $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

Differentiating both eqns. w.r.t. t, we have

$$\frac{dx}{dt} = a\left(-\sin t + \frac{d}{dt}t\sin t\right) \text{ and } \frac{dy}{dt} = a\left(\cos t - \frac{d}{dt}(t\cos t)\right)$$

$$= a\left(-\sin t + t\frac{d}{dt}\sin t + \sin t\frac{d}{dt}t\right)$$

and $$\frac{dy}{dt} = a\left(\cos t - \left(t\frac{d}{dt}(\cos t) + \cos t\frac{d}{dt}(t)\right)\right)$$

$\Rightarrow$ $$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$$

and $$\frac{dy}{dt} = a(\cos t - (-t\sin t + \cos t))$$

$\Rightarrow$ $$\frac{dx}{dt} = at\cos t \quad ...(i)$$

and $$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$$

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \frac{\sin t}{\cos t} = \tan t$

Now differentiating both sides w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \sec^2 t\,\frac{d}{dx}(t) \quad \rightarrow \textbf{Note}$$

$$= \sec^2 t\,\frac{dt}{dx} = \sec^2 t\,\frac{1}{at\cos t} \quad \text{(By } (i))$$

$$= \sec^2 t\,.\,\frac{\sec t}{at} = \frac{\sec^3 t}{at}.$$

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

Sol. Given: $f(x) = |x|^3 = x^3$ if $x \geq 0$ ...(i) [$\because$ $|x| = x$ if $x \geq 0$]

and $f(x) = |x|^3 = (-x)^3 = -x^3$ if $x < 0$ ...(*ii*)

[$\because$ $|x| = -x$ if $x < 0$]

Differentiating both eqns. (*i*) and (*ii*) w.r.t. x,

$f'(x) = 3x^2$ if $x > 0$ and $f'(x) = -3x^2$ if $x < 0$ ...(*iii*)

(At $x = 0$, we can't write the value of $f'(x)$ by usual rule of derivatives because $x = 0$ is a partitioning point of values of $f(x)$ given by (*i*) and (*ii*)

$\therefore \quad f''(x) = 6x$ if $x > 0$ and $= -6x$ if $x < 0$ ...(*iv*)

$\therefore$ From (*iv*), $f''(x)$ exists for all $x > 0$ and for all $x < 0$

i.e., for all $x \in R$ except at $x = 0$ ...(*v*)

Let us discuss derivability of $f(x)$ at $x = 0$

$$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x^3 - 0}{x} \quad \text{[By (ii) and (i)]}$$

$$= \lim_{x \to 0^-} -x^2 = 0 \quad \text{(On putting } x = 0\text{)}$$

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^3 - 0}{x - 0} \quad \text{[By (i)]}$$

$$= \lim_{x \to 0^+} x^2 = 0 \quad \text{(On putting } x = 0\text{)}$$

$\therefore \quad Lf'(0) = Rf'(0) = 0$

$\therefore \quad f(x)$ is derivable at $x = 0$ and $f'(0) = 0$ ...(*vi*)

Let us discuss derivability of $f'(x)$ at $x = 0$

$$Lf''(0) = \lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^-} \frac{-3x^2 - 0}{x}$$

(By (*iii*) and (*vi*))

$$= \lim_{x \to 0^-} (-3x) = -3(0) = 0 \quad \text{(On putting } x = 0\text{)}$$

$$Rf''(0) = \lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \frac{3x^2 - 0}{x} \quad \text{(By (iii) and (vi))}$$

$$= \lim_{x \to 0^+} 3x = 3(0) = 0 \quad \text{(On putting } x = 0\text{)}$$

$\therefore \quad Lf''(0) = Rf''(0) = 0$

$\therefore \quad f'(x)$ is derivable at $x = 0$ and $f''(0) = 0$ ...(*vii*)

From (*iv*) and (*vii*), $f''(x)$ exists for all real x and $f''(x) = 6x$ if $x > 0$ and $= -6x$ if $x < 0$ and $f''(0) = 0$.

19. Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Sol. Let P(n): $\frac{d}{dx}(x^n) = nx^{n-1}$

then P(1): $\frac{d}{dx}(x^1) = 1x^0$ or $\frac{d}{dx}(x) = 1$ which is true.

$\Rightarrow$ P(1) is true.

Assume P(k) is true. *i.e.,* let $\frac{d}{dx}(x^k) = kx^{k-1}$ ...(i)

Now $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k \cdot x) = \frac{d}{dx}(x^k) \cdot x + x^k \cdot \frac{d}{dx}(x)$

$$\left[\because \frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}\right]$$

$= kx^{k-1} \cdot x + x^k \cdot 1$ [Using (i)]

$$= kx^k + x^k = (k + 1)\, x^k \qquad \Rightarrow \text{P}(k + 1) \text{ is true.}$$

Hence by P.M.I., the statement is true for all positive integers n.

20. Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Sol. Given. sin (A + B) = sin A cos B + cos A sin B

Assuming A and B are functions of x and differentiating both sides w.r.t. x, we have

$$\cos (A + B) \cdot \frac{d}{dx}(A + B) = \left[\frac{d}{dx}(\sin A) \cdot \cos B + \sin A \cdot \frac{d}{dx}(\cos B)\right] + \left[\frac{d}{dx}(\cos A) \cdot \sin B + \cos A \cdot \frac{d}{dx}(\sin B)\right]$$

$$\Rightarrow \cos (A + B)\left(\frac{dA}{dx} + \frac{dB}{dx}\right) = \cos A \cdot \frac{dA}{dx} \cdot \cos B + \sin A\,(-\sin B)\frac{dB}{dx} - \sin A \frac{dA}{dx} \cdot \sin B + \cos A \cdot \cos B \frac{dB}{dx}$$

$$= (\cos A \cos B - \sin A \sin B)\frac{dA}{dx} + (\cos A \cos B - \sin A \sin B)\frac{dB}{dx}$$

$$\text{or } \cos (A + B)\left(\frac{dA}{dx} + \frac{dB}{dx}\right) = (\cos A \cos B - \sin A \sin B)\left(\frac{dA}{dx} + \frac{dB}{dx}\right)$$

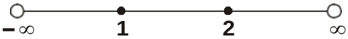
Dividing both sides by $\frac{dA}{dx} + \frac{dB}{dx}$, we have

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

which is the sum formula for cosines.

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points?

Sol. Yes, there exist such function(s).



For example, let us take $f(x) = |x - 1| + |x - 2|$ ...(i)

Let us put each expression within modulus equal to 0 *i.e.,* $x - 1 = 0$ and $x - 2 = 0$ *i.e.,* $x = 1$ and $x = 2$.

These two real numbers $x = 1$ and $x = 2$ divide the whole real line $(-\infty, \infty)$ into three sub-intervals $(-\infty, 1]$, $[1, 2]$ and $[2, \infty)$.

In $(-\infty, 1]$ *i.e.,* **For $x \le 1$,** $x - 1 \le 0$ and $x - 2 \le 0$ and therefore $|x - 1| = -(x - 1)$ and $|x - 2| = -(x - 2)$

$\therefore$ From (i), $f(x) = -(x-1)-(x-2)$

$= -x+1-x+2 = 3-2x$ for $x \le 1$ ...(ii)

In $[1, 2]$ *i.e.,* for $1 \le x \le 2$, $x-1 \ge 0$ and $x-2 \le 0$ and

therefore $|x-1| = x-1$ and $|x-2| = -(x-2)$.
From (i), $f(x) = x - 1 - (x-2) = x - 1 - x + 2 = 1$ for
$1 \le x \le 2$ ...(iii)
Again in $[2, \infty)$ *i.e.,* for $x \ge 2$, $x - 1 \ge 0$ and $x - 2 \ge 0$ and therefore
$|x-1| = x-1$ and $|x-2| = x-2$.
$\therefore$ From (i) $f(x) = x - 1 + x - 2 = 2x - 3$ for $x \ge 2$ ...(iv)
Hence function (i) given in modulus form can be expressed as piece-wise function given by (ii), (iii) and (iv)

i.e., $f(x) = 3 - 2x$ for $x \le 1$ ...(ii)
$= 1$ for $1 \le x \le 2$ ...(iii)
$= 2x - 3$ for $x \ge 2$ ...(iv)

Now the three values of $f(x)$ given by (ii), (iii) and (iv) are polynomial functions and constant function and hence are continuous and derivable for all real values of x except possibly at the partitioning points $x = 1$ and $x = 2$. ....(v)

To examine continuity at $x = 1$

Left Hand limit $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (3 - 2x)$ [By (ii)]

Put $x = 1$; $= 3 - 2 = 1$

Right Hand Limit $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 1$ [By (iii)]

Put $x = 1$; $= 1$

$\therefore \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) \ (= 1)$

$\therefore \lim\limits_{x \to 1} f(x)$ exists and $= 1 = f(1)$ ($\because$ From (iii) $f(1) = 1$]

$\therefore f(x)$ is continuous at $x = 1$ ...(vi)

To examine derivability at $x = 1$

Left Hand derivative $= Lf'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim\limits_{x \to 1^-} \dfrac{3 - 2x - 1}{x - 1}$ [By (ii) and $f(1) = 1$ (proved above)]

$= \lim\limits_{x \to 1^-} \dfrac{-2x + 2}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{-2(x-1)}{x - 1}$

$= \lim\limits_{x \to 1^-} (-2) = -2$

Right Hand derivative $= Rf'(1) = \lim\limits_{x \to 1^+}$

$\underline{f(x) - f(1)}x - 1$

$$= \lim_{x \to 1^+} \frac{1-1}{x-1} \qquad \text{(By } (iii))$$

$$= \lim_{x \to 1^+} \frac{0}{x-1} = \lim_{x \to 1^+} \frac{0}{\text{Non-zero}}$$

$$[x \to 1^+ \Rightarrow x > 1 \Rightarrow x - 1 > 0 \Rightarrow x - 1 \neq 0]$$

$= \lim_{x \to 1^+} 0 = 0$

$\therefore$ $Lf'(1) \neq Rf'(1$

$\therefore$ $f(x)$ is not differentiable at $x = 1$ ...(*vii*)

To examine continuity at $x = 2$

Left hand limit $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 1$ (By (*ii*)) = 1

Right Hand Limit $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3)$ [By (*iv*)]

Putting $x = 2$, $= 4 - 3 = 1$

$\therefore$ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ $(= 1)$

$\therefore$ $\lim_{x \to 2} f(x)$ exists and $= 1 = f(2)$ [$\because$ From (*iii*), $f(2) = 1$]

$\therefore$ $f(x)$ is continuous at $x = 2$ ...(*viii*)

To examine derivability at $x = 2$

$$Lf'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{1 - 1}{x - 2} \quad \text{(By (iii))}$$

$$= \lim_{x \to 2^-} \frac{0}{\text{Non-zero}}$$

[$\because$ $x \to 2^- \Rightarrow x < 2 \Rightarrow x - 2 < 0 \Rightarrow x - 2 \neq 0$]

$$= \lim_{x \to 2^-} 0 = 0$$

$$Rf'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{2x - 3 - 1}{x - 2} \quad \text{[By (iv)]}$$

$$= \lim_{x \to 2^+} \frac{2x - 4}{x - 2} = \lim_{x \to 2^+} \frac{2(x - 2)}{x - 2} = \lim_{x \to 2^+} 2 = 2$$

$\therefore$ $Lf'(2) \neq Rf'(2)$

$\therefore$ $f(x)$ is not differentiable at $x = 2$ ...(*ix*)

From (*v*), (*vi*) and (*viii*), we can say that $f(x)$ is continuous for all real values of x *i.e.,* continuous everywhere.

From (*v*), (*vii*) and (*ix*), we can say that $f(x)$ is not differentiable at exactly two points $x = 1$ and $x = 2$ on the real line.

22. **If** $y =$ | **dx =** | ***f (x) g(x) h(x)l m n a b c***

$\frac{dy}{}$ | ***f′(x) g′(x) h′(x)l m n***

a ***b*** ***c***

, prove that

.

Sol. Given: $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Expanding the determinant along first row,

$$y = f(x)\,(mc - nb) - g(x)\,(lc - na) + h(x)\,(lb - ma)$$

$$\therefore \quad \frac{dy}{dx} = (mc - nb)\frac{d}{dx}f(x) - (lc - na)\frac{d}{dx}g(x) + (lb - ma)\frac{d}{dx}h(x)$$

$$= (mc - nb)\,f'(x) - (lc - na)\,g'(x) + (lb - ma)\,h'(x) \qquad ...(i)$$

$$\text{R.H.S.} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Expanding along first row,

$$= f'(x)\,(mc - nb) - g'(x)\,(lc - na) + h'(x)\,(lb - ma)$$

$$= (mc - nb)\,f'(x) - (lc - na)\,g'(x) + (lb - ma)\,h'(x) \qquad ...(ii)$$

From (i) and (ii), we have L.H.S. = R.H.S.

23. If $y = e^{a\cos^{-1}x}$, $-1 \le x \le 1$, show that

$$\mathbf{(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.}$$

Sol. Given: $y = e^{a\cos^{-1}x}$ ...(i)

$$\therefore \quad \frac{dy}{dx} = e^{a\cos^{-1}x}\frac{d}{dx}(a\cos^{-1}x) \qquad \left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$$

$$\text{or} \quad \frac{dy}{dx} = e^{a\cos^{-1}x}\,.\,a\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-ae^{a\cos^{-1}x}}{\sqrt{1-x^2}}$$

Cross-multiplying, $\sqrt{1-x^2}\,\dfrac{dy}{dx} = -\,a\,e^{a\cos^{-1}x} = -\,ay$

(By (i)) ...(ii)

Again differentiating both sides w.r.t. x,

$$\sqrt{1-x^2}\,\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1 - x^2)^{1/2} = -\,a\frac{dy}{dx}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\,\frac{1}{2}(1 - x^2)^{-1/2}\frac{d}{dx}(1 - x^2) = -\,a\frac{dy}{dx}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{1}{2}\,\frac{dy}{dx}\,\frac{1}{\sqrt{1-x^2}}(-2x) = -a\,\frac{dy}{dx}$$

Multiplying by L.C.M. $= \sqrt{1-x^2}$,

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\sqrt{1-x^2}\,\frac{dy}{dx}$$

$$= -a(-ay) \qquad \text{[By } (ii)\text{]}$$

$$= a^2y$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$