Exercise 2.1

Find the principal values of the following:

1. $\sin^{-1}\left(-\frac{1}{2}\right)$.

Sol. Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$, then $\sin y = -\frac{1}{2}$

Since the range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

therefore, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *i.e.,* y is in fourth quadrant (– θ) or in first quadrant. Also $\sin y$ is negative, therefore, y lies in fourth quadrant and y is negative (*i.e.,* – θ).

Now $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\frac{1}{2}$ $(\because \sin^{-1}(-x) = -\sin^{-1}x)$

$= -\sin^{-1}\sin\frac{\pi}{6} = -\frac{\pi}{6}$

$\therefore$ Principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $\left(-\frac{\pi}{6}\right)$.

2. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Sol. Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$, then $\cos y = \frac{\sqrt{3}}{2}$

Since the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$, therefore, $y \in [0, \pi]$ *i.e.,* y is in first or second quadrant. Also $\cos y$ is positive, therefore, y lies in first quadrant.

Now $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \cos^{-1}\cos\frac{\pi}{6} = \frac{\pi}{6}$

$\therefore$ Principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

3. cosec^{-1} **(2).**

Sol. Let $\theta = \text{cosec}^{-1} 2$ $\therefore$ θ is in first quadrant because $x = 2 > 0$.
($\because$ If $x > 0$, then value of each inverse function lies in first quadrant.)

$\therefore$ $\theta = \text{cosec}^{-1} 2 = \text{cosec}^{-1} \text{cosec} \frac{\pi}{6} = \frac{\pi}{6}$.

4. $\tan^{-1} (-\sqrt{3})$.

Sol. Let $\tan^{-1} (-\sqrt{3}) = y$, then $\tan y = -\sqrt{3}$

Since the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, therefore, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *i.e.,* y is in fourth quadrant

$(-\theta)$ or y is in first quadrant. Also $\tan y$ is negative, therefore, y lies in fourth quadrant and y is negative (*i.e.,* $-\theta$).

Now $\tan^{-1}(-\sqrt{3}) = -\tan^{-1}\sqrt{3}$ $\quad (\because \tan^{-1}(-x) = -\tan^{-1}x)$

$= -\tan^{-1}\tan\frac{\pi}{3} = -\frac{\pi}{3}$

$\therefore$ Principal value of $\tan^{-1}(-\sqrt{3})$ is $\left(-\frac{\pi}{3}\right)$.

5. $\cos^{-1}\left(-\frac{1}{2}\right)$.

Sol. Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$, then $\cos y = -\frac{1}{2}$

Since the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$, therefore, $y \in [0, \pi]$ *i.e.,* y is in first or second quadrant. Also $\cos y$ is negative, therefore, y lies in second quadrant (*i.e.,* $y = \pi - \theta$).

Now $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2}$ $\quad (\because \cos^{-1}(-x) = \pi - \cos^{-1}x)$

$= \pi - \cos^{-1}\cos\frac{\pi}{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore$ Principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

6. $\tan^{-1}(-1)$.

Sol. Let $\theta = \tan^{-1}(-1)$ $\therefore$ θ lies between $-\frac{\pi}{2}$ and 0 $(\because x = -1 < 0)$

[**Note.** For $x < 0$, values of $\sin^{-1} x$, $\tan^{-1} x$ and $\text{cosec}^{-1} x$ lies between $-\frac{\pi}{2}$ and 0.]

$\therefore \tan^{-1}(-1) = -\tan^{-1}1 = -\tan^{-1}\tan\frac{\pi}{4} = -\frac{\pi}{4}$

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

Sol. Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$, then $\sec y = \frac{2}{\sqrt{3}}$

Since the range of the principal value branch of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$, therefore, $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$ *i.e.,* y is in first quadrant or second quadrant. Also sec y is positive, therefore, y lies in first quadrant.

Now, $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \sec^{-1}\left(\sec\frac{\pi}{6}\right) = \frac{\pi}{6}$

$\therefore$ Principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

8. $\cot^{-1}(\sqrt{3})$.

Sol. Let $\theta = \cot^{-1}(\sqrt{3})$

$\therefore$ θ is in first quadrant because $x = \sqrt{3} > 0$.

$\therefore$ $\theta = \cot^{-1}\sqrt{3} = \cot^{-1}\cot\frac{\pi}{6} = \frac{\pi}{6}$.

9. $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$.

Sol. Let $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$\therefore$ θ lies between $\frac{\pi}{2}$ and π $\left(\because\ x = -\frac{1}{2} < 0\right)$

(**Note.** For $x < 0$, value of $\cos^{-1} x$, $\cot^{-1} x$ and $\sec^{-1} x$ lies between $\frac{\pi}{2}$ and π.)

$\therefore$ $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\frac{1}{\sqrt{2}}$

$= \pi - \cos^{-1}\cos\frac{\pi}{4} = \pi - \frac{\pi}{4} = \frac{4\pi - \pi}{4} = \frac{3\pi}{4}$.

10. $\text{cosec}^{-1}(-\sqrt{2})$.

Sol. Let $\text{cosec}^{-1}(-\sqrt{2}) = y$, then cosec $y = -\sqrt{2}$

Since the range of the principal value branch of cosec^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$- \{0\}$, therefore, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$. Also cosec y is negative,

therefore, y lies in fourth quadrant $(-\theta)$ and y is negative.

Now, $\text{cosec}^{-1}(-\sqrt{2}) = -\text{cosec}^{-1}\sqrt{2}$ $\quad (\because \text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x)$

$$= -\text{cosec}^{-1}\text{cosec}\frac{\pi}{4} = -\frac{\pi}{4}$$

$\therefore$ Principal value of $\text{cosec}^{-1}(-\sqrt{2})$ is $\left(-\frac{\pi}{4}\right)$.

Find the value of the following:

11. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

Sol. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$$= \tan^{-1}1 + \pi - \cos^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{2}$$

$$= \tan^{-1}\tan\frac{\pi}{4} + \pi - \cos^{-1}\cos\frac{\pi}{3} - \sin^{-1}\sin\frac{\pi}{6}$$

$$= \frac{\pi}{4} + \pi - \frac{\pi}{3} - \frac{\pi}{6} = \frac{3\pi + 12\pi - 4\pi - 2\pi}{12}$$

$= \frac{9\pi}{12} = \frac{3\pi}{4}$

12. $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

Sol. $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \cos^{-1}\cos\frac{\pi}{3} + 2\sin^{-1}\sin\frac{\pi}{6}$

$= \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$.

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ **(B)** $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ **(D)** $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Sol. Option (B) is the correct answer.

(By definition of principal value for $y = \sin^{-1} x$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$)

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ **is equal to**

(A) π **(B)** $-\frac{\pi}{3}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{2\pi}{3}$.

Sol. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$= \tan^{-1}\sqrt{3} - (\pi - \sec^{-1} 2)$ $(\because \sec^{-1}(-x) = \pi - \sec^{-1} x)$

$= \tan^{-1}\tan\frac{\pi}{3} - \pi + \sec^{-1}\sec\frac{\pi}{3}$

$= \frac{\pi}{3} - \pi + \frac{\pi}{3} = \frac{\pi - 3\pi + \pi}{3} = -\frac{\pi}{3}$

$\therefore$ Option (B) is the correct answer.

Exercise 2.2

Prove the following:

1. $3\sin^{-1}x = \sin^{-1}(3x - 4x^3),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Sol. To prove: $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Put $\sin\theta = x$ ($\Rightarrow\ \theta = \sin^{-1}x$)

$\therefore\ \sin 3\theta = 3x - 4x^3 \quad \Rightarrow \quad 3\theta = \sin^{-1}(3x - 4x^3)$

Putting $\theta = \sin^{-1}x$, $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$.

2. $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x),\ x \in \left[\frac{1}{2}, 1\right]$.

Sol. To prove: $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x),\ x \in \left[\frac{1}{2}, 1\right]$

Let $\cos^{-1}x = \theta$, then $x = \cos\theta$

We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$

$\Rightarrow 3\theta = \cos^{-1}(4x^3 - 3x) \Rightarrow \quad 3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$.

3. $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$.

Sol. To prove: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$$\text{L.H.S.} = \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1 - \frac{2}{11}\times\frac{7}{24}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{48+77}{264-14} = \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2} = \text{R.H.S.}$$

4. $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$.

Sol. To prove: $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

$$\text{L.H.S.} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \quad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{28+3}{21-4} = \tan^{-1}\frac{31}{17} = \text{R.H.S.}$$

Write the following functions in the simplest form:

5. $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $x \neq 0$.

Sol. Put $x = \tan\theta$ so that $\theta = \tan^{-1}x$

$$\therefore \quad \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x.$$

6. $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}}$**, $|x| > 1$.**

Sol. To simplify $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}}$, put $x = \sec\theta$ (See Note (*iii*) below) ($\Rightarrow \theta = \sec^{-1}x$)

$$= \tan^{-1}\frac{1}{\sqrt{\sec^2\theta - 1}} = \tan^{-1}\left(\frac{1}{\sqrt{\tan^2\theta}}\right)$$

$| \because \sec^2\theta - \tan^2\theta = 1 \Rightarrow \sec^2\theta - 1 = \tan^2\theta$

$$= \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \tan^{-1}(\cot\theta)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1}x.$$

Very useful Note: (*i*) For $\sqrt{a^2 - x^2}$, put $x = a\sin\theta$

(*ii*) For $\sqrt{a^2 + x^2}$, put $x = a\tan\theta$

and (*iii*) For $\sqrt{x^2 - a^2}$, put $x = a\sec\theta$.

7. $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$**, $x < \pi$.**

Sol. $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}} = \tan^{-1}\sqrt{\dfrac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}$

$[\because\ 1 - \cos 2\theta = 2\sin^2\theta \text{ and } 1 + \cos 2\theta = 2\cos^2\theta]$

$= \tan^{-1}\sqrt{\tan^2\frac{x}{2}} = \tan^{-1}\tan\frac{x}{2} = \frac{x}{2}$.

8. $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$**, $0 < x < \pi$.**

Sol. The given expression $= \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Dividing the numerator and denominator by $\cos x$,

$$= \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right) = \tan^{-1}\tan\left(\frac{\pi}{4} - x\right)$$

$$= \frac{\pi}{4} - x.$$

9. $\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}, \; |x| < a.$

Sol. To simplify $\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}$, put $x = a\sin\theta$;

(See note (*i*) below solution of Q. No. 7)

$$= \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2(1 - \sin^2\theta)}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2\cos^2\theta}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a}$$

$$\left[\because\; x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}\right]$$

10. $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), \; a > 0, \; \left(-\frac{a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}\right).$

Sol. $\tan^{-1}\left\{\frac{3a^2x - x^3}{a^3 - 3ax^2}\right\}$

(Dividing the numerator and denominator by a^3, to make the first term in denominator as 1)

$$= \tan^{-1}\left(\frac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{\quad}\right.$$

$$\left(1 - 3\left(\frac{x}{a}\right)^2 \right)$$

Put $\frac{x}{a} = \tan\theta$.

$\therefore$ The given expression $= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}\frac{x}{a}.$$

Find the values of each of the following:

11. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$.

Sol. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right] = \tan^{-1}\left[2\cos\left(2\sin^{-1}\sin\frac{\pi}{6}\right)\right]$

$$= \tan^{-1}\left[2\cos\left(2 . \frac{\pi}{6}\right)\right] = \tan^{-1}\left[2\cos\frac{\pi}{3}\right]$$

$$= \tan^{-1}\left(2 \times \frac{1}{2}\right) = \tan^{-1} 1 = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

12. $\cot(\tan^{-1} a + \cot^{-1} a)$

Sol. $\cot(\tan^{-1} a + \cot^{-1} a)$

$$= \cot\frac{\pi}{2} = 0. \qquad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

13. $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$, $|x| < 1$, $y > 0$ and $xy < 1$.

Sol. Put $x = \tan\theta$ and $y = \tan\phi$, then the given expression

$$= \tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right)$$

$$= \tan\left(\frac{1}{2}\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} + \frac{1}{2}\cos^{-1}\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)$$

$$= \tan\left[\frac{1}{2}\sin^{-1}(\sin 2\theta) + \frac{1}{2}\cos^{-1}(\cos 2\phi)\right]$$

$$= \tan\left[\frac{1}{2}(2\theta) + \frac{1}{2}(2\phi)\right] = \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{x+y}{1-xy}.$$

14. **If** $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$, **then find the value of** x.

Sol. **Given :** $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = \sin\frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos^{-1} x = \frac{\pi}{2} - \sin^{-1}\frac{1}{5} = \cos^{-1}\frac{1}{5} \quad \left(\because \sin^{-1} t + \cos^{-1} t = \frac{\pi}{2}\right)$$

$$\Rightarrow \quad x = \frac{1}{5}.$$

15. If $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$, then find the value of x.

Sol. Given: $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \dfrac{\pi}{4}$

$$\Rightarrow \tan^{-1}\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}=\frac{\pi}{4}\qquad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right)$$

Multiplying by L.C.M. = $(x - 2)(x + 2)$,

$$\Rightarrow \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x^2+2x-x-2+x^2-2x+x-2}{x^2-4-(x^2-1)} = 1$$

$$\Rightarrow \frac{2x^2-4}{x^2-4-x^2+1} = 1 \Rightarrow \frac{2x^2-4}{-3} = 1$$

$$\Rightarrow 2x^2-4=-3 \Rightarrow 2x^2=4-3=1$$

$$\Rightarrow x^2=\frac{1}{2} \quad \therefore x=\pm\sqrt{\frac{1}{2}}=\pm\frac{1}{\sqrt{2}}.$$

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

Sol. We know that $\sin^{-1}(\sin x) = x$. Therefore, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\frac{2\pi}{3}$.

But $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ which is the principal value branch of $\sin^{-1}$.

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left(\sin\frac{3\pi-\pi}{3}\right)=\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]$

$=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$ and $\frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ $\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\frac{\pi}{3}$.

17. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Sol. We know that $\tan^{-1}(\tan x) = x$. Therefore, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\frac{3\pi}{4}$.

But $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ which is the principal value branch of $\tan^{-1}$

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\frac{4\pi-\pi}{4}\right) = \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{4}\right)\right]$

$= \tan^{-1}\left[-\tan\frac{\pi}{4}\right] = -\tan^{-1}\tan\frac{\pi}{4}$

$= -\frac{\pi}{4}$ and $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\therefore$ $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$.

18. $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$.

Sol. Let $\sin^{-1}\frac{3}{5} = x$ and $\cot^{-1}\frac{3}{5} = y$

$\Rightarrow x$ and y both lie in first quadrant because $\frac{3}{5} > 0$ and also $\frac{3}{2} > 0$

and hence $\cos x$ must be positive.

and $\sin x = \frac{3}{5}$ and $\cot y = \frac{3}{2}$

$\Rightarrow \quad \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\Rightarrow \quad \tan x = \frac{\sin x}{\cos x} = \frac{3}{4}$ and $\tan y = \frac{2}{3}$

$\therefore \quad \tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \tan(x+y)$

$= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}} = \frac{\frac{17}{12}}{\frac{1}{2}} = \frac{17}{6}$.

19. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ **is equal to**

(A) $\frac{7\pi}{6}$ **(B)** $\frac{5\pi}{6}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{6}$.

Sol. We know that $(x =) \cos\frac{7\pi}{6} = \cos\left(7\times\frac{180^\circ}{6}\right) = \cos 210^\circ$ is negative. ($\because$ 210° lies in third quadrant)

$\therefore$ Value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ must lie between $\frac{\pi}{2}$ and π.

$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{7\pi}{6}\right)\right) \quad |\because \cos(2\pi-\theta) = \cos\theta$

$= 2\pi - \frac{7\pi}{6} = \frac{12\pi - 7\pi}{6} = \frac{5\pi}{6}$

$\therefore$ Option (B) is the correct answer.

20. $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ **is equal to**

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** 1.

Sol. $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \sin^{-1}\frac{1}{2}\right)$ $\because \sin^{-1}(-x) = -\sin^{-1} x$

$= \sin\left(\frac{\pi}{3} + \sin^{-1}\left(\sin\frac{\pi}{6}\right)\right)$

$= \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\left(\frac{2\pi+\pi}{6}\right) = \sin\frac{3\pi}{6} = \sin\frac{\pi}{2} = 1.$

∴ Option (D) is the correct answer.

21. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ **is equal to**

(A) π **(B)** $-\frac{\pi}{2}$ **(C)** 0 **(D)** $2\sqrt{3}$.

Sol. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$

$\tan^{-1}\sqrt{3} - (\pi - \cot^{-1}\sqrt{3})$ $\because \cot^{-1}(-x) = \pi - \cot^{-1}x$

$\tan^{-1}\tan\frac{\pi}{3} - \left(\pi - \cot^{-1}\left(\cot\frac{\pi}{6}\right)\right)$

$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{6}\right) = \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6}$

$= -\frac{3\pi}{6} = -\frac{\pi}{2}$. ∴ Option (B) is the correct answer.

MISCELLANEOUS EXERCISE

Find the value of the following:

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Sol. Here $(x) = \cos\frac{13\pi}{6} = \cos\frac{12\pi+\pi}{6} = \cos\left(2\pi+\frac{\pi}{6}\right)$

$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} > 0.$

$\therefore$ Value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ lies in first quadrant.

$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\frac{\sqrt{3}}{2} = \cos^{-1}\cos\frac{\pi}{6} = \frac{\pi}{6}.$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$.

Sol. Here $(x) = \tan\frac{7\pi}{6} = \tan\frac{6\pi+\pi}{6} = \tan\left(\pi+\frac{\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} > 0$

$\therefore$ Value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ lies in first quadrant.

$\therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\frac{1}{\sqrt{3}} = \tan^{-1}\tan\frac{\pi}{6} = \frac{\pi}{6}.$

3. Prove that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$.

Sol. Let $\sin^{-1}\frac{3}{5} = \theta$

$\Rightarrow$ θ lies in first quadrant $\left(\because \frac{3}{5} > 0\right)$ and $\sin\theta = \frac{3}{5}$.

$\therefore$ $\cos\theta$ is positive and $= \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\therefore$ $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

We know that $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times\frac{3}{4}}{1-\frac{9}{16}}$

or $\tan 2\theta = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2}\times\frac{16}{7} = \frac{24}{7}$ or $2\theta = \tan^{-1}\frac{24}{7}$

Putting $\theta = \sin^{-1}\frac{3}{5}$, $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$.

4. Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$.

Sol. Let $\sin^{-1}\frac{8}{17} = \alpha$ $\Rightarrow$ α is in first quadrant. $\left(\because \frac{8}{17} > 0\right)$

and $\sin\alpha = \frac{8}{17}$

$\therefore$ $\cos\alpha$ is positive and $= \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{64}{289}}$

$= \sqrt{\frac{289-64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$

$\therefore$ $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}$

Again let $\sin^{-1}\frac{3}{5} = \beta$ $\Rightarrow \beta$ is in first quadrant. $\left(\because \frac{3}{5} > 0\right)$

and $\sin\beta = \frac{3}{5}$

$\therefore$ cos β is also positive and $= \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$$\therefore \quad \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

We know that **tan (α + β)** = $\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Putting values of tan α and tan β, = $\dfrac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15}\cdot\frac{3}{4}}$

Multiplying by L.C.M. = 60, $= \dfrac{32 + 45}{60 - 24} = \dfrac{77}{36}$

i.e., $\tan(\alpha + \beta) = \dfrac{77}{36}$

∴ $\alpha + \beta = \tan^{-1}\dfrac{77}{36}$

Putting values of α and β, $\sin^{-1}\dfrac{8}{17} + \sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{77}{36}$.

5. Prove that $\cos^{-1}\dfrac{4}{5} + \cos^{-1}\dfrac{12}{13} = \cos^{-1}\dfrac{33}{65}$.

Sol. Let $\cos^{-1}\dfrac{4}{5} = \alpha$ ⇒ α is in first quadrant. $\left(\because \dfrac{4}{5} > 0\right)$

and $\cos\alpha = \dfrac{4}{5}$

∴ sin α is also positive and $= \sqrt{1 - \cos^2\alpha}$

$= \sqrt{1 - \dfrac{16}{25}} = \sqrt{\dfrac{9}{25}} = \dfrac{3}{5}$

Again let $\cos^{-1}\dfrac{12}{13} = \beta$

⇒ β is in first quadrant. $\left(\because \dfrac{12}{13} > 0\right)$

and $\cos\beta = \dfrac{12}{13}$.

∴ sin β is also positive and $= \sqrt{1 - \cos^2\beta}$

$= \sqrt{1 - \left(\dfrac{12}{13}\right)^2} = \sqrt{1 - \dfrac{144}{169}} = \sqrt{\dfrac{169 - 144}{169}} = \sqrt{\dfrac{25}{169}} = \dfrac{5}{13}$

We know that **cos (α + β)** = cos α cos β − sin α sin β

Putting values, $= \dfrac{4}{5}\left(\dfrac{12}{13}\right) - \dfrac{3}{5}\left(\dfrac{5}{13}\right)$

or $$\cos(\alpha + \beta) = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

$\therefore$ $$\alpha + \beta = \cos^{-1}\frac{33}{65}$$

Putting values of α and β, $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$.

6. **Prove that $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$.**

Sol. Let $\cos^{-1}\frac{12}{13} = \alpha \Rightarrow \alpha$ is in first quadrant. $\left(\because \frac{12}{13} > 0\right)$

and $\cos\alpha = \frac{12}{13}$.

∴ sin α is also positive and $= \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{144}{169}}$

$$= \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

Let $\sin^{-1}\frac{3}{5} = \beta \Rightarrow \beta$ is in first quadrant. $\left(\because \frac{3}{5} > 0\right)$

and $\sin\beta = \frac{3}{5}$.

∴ cos β is also positive and $= \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{9}{25}}$

$$= \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

We know that **sin (α + β)** = sin α cos β + cos α sin β.

Putting values, $\sin(\alpha+\beta) = \frac{5}{13}\left(\frac{4}{5}\right) + \frac{12}{13}\left(\frac{3}{5}\right) = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$

$$\therefore \quad \alpha + \beta = \sin^{-1}\frac{56}{65}$$

Putting values of α and β, $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$.

7. **Prove that $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$.**

Sol. Let $\sin^{-1}\frac{5}{13} = x$ and $\cos^{-1}\frac{3}{5} = y$

$\Rightarrow$ x and y both lie in first quadrant because $\frac{5}{13} > 0$ and $\frac{3}{5} > 0$

and hence cos x and sin y are both positive

and $\sin x = \frac{5}{13}$ and $\cos y = \frac{3}{5}$

$\Rightarrow$ $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

and $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$$\Rightarrow \quad \tan x = \frac{\sin x}{\cos x} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$

and $$\tan y = \frac{\sin y}{\cos y} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

Now, $$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}}$$

$$= \frac{\frac{21}{12}}{\frac{4}{9}} = \frac{7}{4} \times \frac{9}{4} = \frac{63}{16}$$

$$\Rightarrow \quad \tan^{-1}\frac{63}{16} = x + y$$

Putting values of x and y, $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$.

8. Prove that $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

Sol. L.H.S. $= \left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1 - \frac{1}{5}\cdot\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1 - \frac{1}{3}\cdot\frac{1}{8}}\right)$$

$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy} \text{ if } x > 0, y > 0, \text{ and } xy < 1.\right]$

Here for first sum, $xy = \frac{1}{5} \times \frac{1}{7} = \frac{1}{35} < 1$ and for second sum

$$xy = \frac{1}{3} \times \frac{1}{8} = \frac{1}{24} < 1. \Big]$$

$$= \tan^{-1}\left(\frac{\frac{7+5}{35}}{\frac{35-1}{35}}\right) + \tan^{-1}\left(\frac{\frac{8+3}{24}}{\frac{24-1}{24}}\right) = \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$= \tan^{-1}\left(\dfrac{\dfrac{6}{17}+\dfrac{11}{23}}{1-\dfrac{6}{17}\cdot\dfrac{11}{23}}\right)$ $\left[\because xy = \dfrac{6}{17}\times\dfrac{11}{23} = \dfrac{66}{391} < 1\right]$

Multiplying NUM and DEN by 17×23

$= \tan^{-1}\left(\dfrac{138+187}{391-66}\right) = \tan^{-1}\left(\dfrac{325}{325}\right)$

$= \tan^{-1} 1 = \tan^{-1} \tan\dfrac{\pi}{4} = \dfrac{\pi}{4}$ = R.H.S.

9. Prove that $\tan^{-1}\sqrt{x} = \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x}{1+x}\right), x \in [0, 1]$.

Sol. Let $\tan^{-1}\sqrt{x} = \theta$, then $\sqrt{x} = \tan\theta$ $\therefore x = \tan^2\theta$

$\therefore$ R.H.S. $= \dfrac{1}{2}\cos^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\cos^{-1}\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$= \dfrac{1}{2}\cos^{-1}(\cos 2\theta) = \dfrac{1}{2}(2\theta) = \theta = \tan^{-1}\sqrt{x}$ = L.H.S.

10. Prove that $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \dfrac{x}{2}, x \in \left(0, \dfrac{\pi}{4}\right)$.

Sol. We know that

$1 + \sin x = \cos^2\dfrac{x}{2} + \sin^2\dfrac{x}{2} + 2\cos\dfrac{x}{2}\sin\dfrac{x}{2} = \left(\cos\dfrac{x}{2}+\sin\dfrac{x}{2}\right)^2$

Similarly, $1 - \sin x = \left(\cos\dfrac{x}{2} - \sin\dfrac{x}{2}\right)^2$

$\therefore \cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$$= \cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2}.$$

11. Prove that $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1} x$, $\dfrac{-1}{\sqrt{2}} \le x \le 1$.

Sol. L.H.S. $= \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Put $x = \cos 2\theta$ $\left(\Rightarrow 2\theta = \cos^{-1} x \Rightarrow \theta = \dfrac{1}{2}\cos^{-1} x\right)$

$\therefore$ L.H.S. $= \tan^{-1}\left(\dfrac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$

Dividing every term in NUM and DEN by $\sqrt{2}\cos\theta$,

$= \tan^{-1}\left(\dfrac{1-\tan\theta}{1+\tan\theta}\right) = \tan^{-1}\left(\dfrac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)$

$= \tan^{-1}\tan\left(\dfrac{\pi}{4}-\theta\right) = \dfrac{\pi}{4} - \theta$

$= \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1} x =$ R.H.S.

12. Prove that $\dfrac{9\pi}{8} - \dfrac{9}{4}\sin^{-1}\dfrac{1}{3} = \dfrac{9}{4}\sin^{-1}\dfrac{2\sqrt{2}}{3}$.

Sol. L.H.S. $= \dfrac{9\pi}{8} - \dfrac{9}{4}\sin^{-1}\dfrac{1}{3}$

$= \dfrac{9}{4}\left(\dfrac{\pi}{2} - \sin^{-1}\dfrac{1}{3}\right)$

$= \dfrac{9}{4}\cos^{-1}\dfrac{1}{3}$ $\left(\because \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2} \Rightarrow \dfrac{\pi}{2} - \sin^{-1} x = \cos^{-1} x\right)$

$\overline{4}$ $\quad \overline{3}$ $\quad \overline{2}$ $\quad 2$

$\Rightarrow$ L.H.S. $= \frac{9}{4}\theta$ ...(*i*) where $\theta = \cos^{-1}\frac{1}{3}$

$\therefore$ θ is in first quadrant $\left(\because \frac{1}{3} > 0\right)$ and $\cos\theta = \frac{1}{3}$

$\therefore$ $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \sqrt{\frac{4\times 2}{9}} = \frac{2}{3}\sqrt{2}$

$\therefore$ $\theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Putting this value of θ in (*i*), L.H.S. = $\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

= R.H.S.

13. Solve the equation 2 tan⁻¹ (cos *x*) = tan⁻¹ (2 cosec *x*).

Sol. The given equation is

$2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$ $\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$

$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$

Dividing both sides by $\frac{2}{\sin x}$, we have $\frac{\cos x}{\sin x} = 1$

$\therefore \cot x = 1 = \cot\frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}$.

14. Solve the equation $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, (x > 0)$.

Sol. Put $x = \tan\theta$

$\therefore$ The given equation becomes $\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$

$\Rightarrow \tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right] = \frac{1}{2}\theta$

$\Rightarrow \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right) = \frac{\theta}{2}$

$\Rightarrow \frac{\pi}{4} - \theta = \frac{\theta}{2} \Rightarrow \theta + \frac{\theta}{2} = \frac{\pi}{4}$

$\Rightarrow \frac{3\theta}{2} = \frac{\pi}{4} \Rightarrow 12\theta = 2\pi \Rightarrow \theta = \frac{2\pi}{12} = \frac{\pi}{6}$

$\therefore \quad x = \tan\theta = \tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}.$

15. sin (tan⁻¹ x), $|x| < 1$ is equal to

(A) $\dfrac{x}{\sqrt{1-x^2}}$ **(B)** $\dfrac{1}{\sqrt{1-x^2}}$ **(C)** $\dfrac{1}{\sqrt{1+x^2}}$ **(D)** $\dfrac{x}{\sqrt{1+x^2}}$.

Sol. $\sin(\tan^{-1} x) = \sin\theta$ where $\theta = \tan^{-1} x$ ($\Rightarrow x = \tan\theta$)

$$= \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}}$$

$[\because \operatorname{cosec}^2\theta - \cot^2\theta = 1 \Rightarrow \operatorname{cosec}^2\theta = 1 + \cot^2\theta]$

Putting $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}$,

$$\sin(\tan^{-1} x) = \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{\frac{x^2+1}{x^2}}} = \frac{x}{\sqrt{x^2+1}}$$

$\therefore$ Option (D) is the correct answer.

16. $\sin^{-1}(1 - x) - 2\sin^{-1} x = \frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ **(B) $1, \frac{1}{2}$** **(C) 0** **(D) $\frac{1}{2}$.**

Sol. The given equation is $\sin^{-1}(1 - x) - 2\sin^{-1} x = \frac{\pi}{2}$ ...(i)

Put $\sin^{-1} x = \theta$ $\therefore$ $x = \sin\theta$ ...(ii)

$\therefore$ Equation (i) becomes $\sin^{-1}(1 - x) - 2\theta = \frac{\pi}{2}$

$\Rightarrow$ $\sin^{-1}(1 - x) = \frac{\pi}{2} + 2\theta$

$\Rightarrow$ $1 - x = \sin\left(\frac{\pi}{2} + 2\theta\right) = \cos 2\theta = 1 - 2\sin^2\theta$

Putting $\sin\theta = x$ from (ii), $1 - x = 1 - 2x^2$

or $-x = -2x^2$ or $2x^2 - x = 0$ or $x(2x - 1) = 0$

$\therefore$ Either $x = 0$ or $2x - 1 = 0$ *i.e.*, $2x = 1$

i.e., $x = \frac{1}{2}$.

Let us test these roots

Putting $x = 0$ in (i), $\sin^{-1} 1 - 2\sin^{-1} 0 = \frac{\pi}{2}$

or $\frac{\pi}{2} - 0 = \frac{\pi}{2}$ or $\frac{\pi}{2} = \frac{\pi}{2}$ which is true.

$\therefore$ $x = 0$ is a root.

Putting $x = \frac{1}{2}$ in (i), $\sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{2} = \frac{\pi}{2}$

or $-\sin^{-1}\frac{1}{2} = \frac{\pi}{2}$ $[\because\ t - 2t = -t]$

or $-\frac{\pi}{6} = \frac{\pi}{2}$ $\left[\because \sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\sin\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6}\right]$ which is impossible.

$\therefore\ x = \frac{1}{2}$ is rejected.

$\therefore$ Option (C) is the correct answer.

17. $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ **is equal to**

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{4}$ **(D)** $-\frac{3\pi}{4}$.

Sol. $\tan^{-1}\frac{x}{y} - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

$$= \tan^{-1}\left[\frac{\frac{x}{y} - \left(\frac{x-y}{x+y}\right)}{1 + \frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right] \quad \left(\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}\right)$$

Multiplying both numerator and denominator by $y(x + y)$

$$= \tan^{-1}\left[\frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}\right] = \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + y^2}\right) = \tan^{-1} 1 = \tan^{-1}\tan\frac{\pi}{4} = \frac{\pi}{4}$$

$\therefore$ Option (C) is the correct answer.