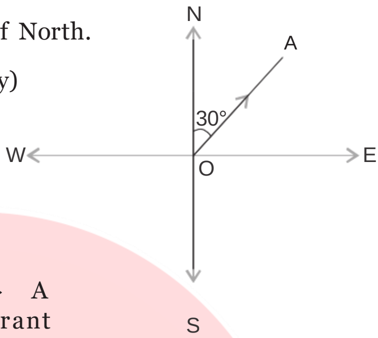
Exercise 10.1

1. Represent graphically a displacement of 40 km, 30° east of north.

Sol. Displacement 40 km, 30° East of North.

$\Rightarrow$ Displacement vector $\overrightarrow{OA}$ (say)

such that $|\overrightarrow{OA}| = 40$ (given) and vector $\overrightarrow{OA}$ makes an angle 30° with North in East-North quadrant.



Note. α° South of West $\Rightarrow$ A vector in South-West quadrant making an angle of α° with West.

2. Check the following measures as scalars and vectors:

(i) 10 kg **(*ii*) 2 meters north-west** **(*iii*) 40°**

(iv) 40 Watt **(*v*) 10^{-19} coulomb** **(*vi*) 20 m/sec².**

Sol. (*i*) 10 kg is a measure of mass and therefore a **scalar.**

($\because$ 10 kg has no direction, it is magnitude only).

(ii) 2 meters North-West is a measure of velocity (*i.e.,* has magnitude and direction both) and hence is a **vector.**

(iii) 40° is a measure of angle *i.e.,* is magnitude only and, therefore, a scalar.

(iv) 40 Watt is a measure of power (*i.e.,* 40 watt has no direction) and, therefore, a scalar.

(v) 10^{-19} coulomb is a measure of electric charge (*i.e.,* is magnitude only) and, therefore, a scalar.

(vi) 20 m/sec² is a measure of acceleration *i.e.,* is a measure of rate of change of velocity and hence is a vector.

3. Classify the following as scalar and vector quantities:

(i) time period **(*ii*) distance** **(*iii*) force**

(*iv*) velocity **(*v*) work done.**

Sol. (*i*) Time-scalar (*ii*) Distance-scalar (*iii*) Force-vector

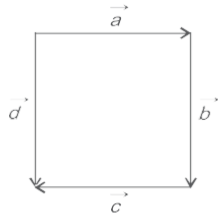
(*iv*) Velocity-vector (*v*) Work done-scalar.

4. In the adjoining figure, (a square), identify the following vectors.

(i) Coinitial

(ii) Equal

(iii) Collinear but not equal.



Sol. (*i*) $\vec{a}$ and $\vec{d}$ have same initial point and,

a
therefore, coinitial vectors.

(*ii*) $\vec{b}$ and $\vec{d}$ have same direction and same magnitude. Therefore, $\vec{b}$ and $\vec{d}$ are equal vectors.

(*iii*) $\vec{a}$ and $\vec{c}$ have parallel supports, so that they are collinear. Since they have opposite directions, they are not equal. Hence $\vec{a}$ and $\vec{c}$ are collinear but not equal.

5. Answer the following as true or false.

(i) $\vec{a}$ and $-\vec{a}$ are collinear.
(ii) Two collinear vectors are always equal in magnitude.
(iii) Two vectors having same magnitude are collinear.
(iv) Two collinear vectors having the same magnitude are equal.

Sol. (*i*) True.
(ii) False. ($\because$ $\vec{a}$ and $2\vec{a}$ are collinear vectors but $|2\vec{a}| = 2|\vec{a}|$)
(iii) False.
($\because$ $|\hat{i}| = |\hat{j}| = 1$ but $\hat{i}$ and $\hat{j}$ are vectors along x-axis (OX) and y-axis (OY) respectively.)
(iv) False.
($\because$ Vectors $\vec{a}$ and $-\vec{a}$ $(= (-1)\vec{a} = m\vec{a})$ are collinear vectors and $|\vec{a}| = |-\vec{a}|$ but we know that $\vec{a} \neq -\vec{a}$ because their directions are opposite).

Note. Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if

(i) $|\vec{a}| = |\vec{b}|$ (*ii*) $\vec{a}$ and $\vec{b}$ have same (like) direction.

Exercise 10.2

1. **Compute the magnitude of the following vectors:**

$$\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k},$$

$$\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}.$$

Sol. Given: $\vec{a} = \hat{i} + \hat{j} + \hat{k}.$

Therefore, $|\vec{a}| = \sqrt{x^2 + y^2 + x^2} = \sqrt{1+1+1} = \sqrt{3}.$

$$\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}.$$

Therefore, $|\vec{b}| = \sqrt{4+49+9} = \sqrt{62}.$

$$\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}.$$

Therefore, $|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{-1}{\sqrt{3}}\right)^2}$

$$= \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = \sqrt{\frac{3}{3}} = \sqrt{1} = 1.$$

2. **Write two different vectors having same magnitude.**

Sol. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

Clearly, $\vec{a} \neq \vec{b}$. ($\because$ Coefficients of $\hat{i}$ and $\hat{j}$ are same in vectors $\vec{a}$ and $\vec{b}$ but coefficients of $\hat{k}$ in $\vec{a}$ and $\vec{b}$ are unequal as $1 \neq -1$).

But $|\vec{a}| = \sqrt{x^2 + y^2 + x^2} = \sqrt{1+1+1} = \sqrt{3}$

and $|\vec{b}| = \sqrt{1+1+1} = \sqrt{3}$ $\therefore$ $|\vec{a}| = |\vec{b}|$.

Remark. In this way, we can construct an infinite number of possible answers.

3. **Write two different vectors having same direction.**

Sol. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ ...(*i*)

and $\vec{b} = 2(\hat{i} + 2\hat{j} + 3\hat{k})$ ...(*ii*)

$= 2\vec{a}$ [By (*i*)]

$\therefore$ $\vec{b} = m\vec{a}$ where $m = 2 > 0$.

$\therefore$ Vectors $\vec{a}$ and $\vec{b}$ have the same direction.

But $\vec{b} \neq \vec{a}$ [$\because$ $\vec{b} = 2\vec{a} \Rightarrow |\vec{b}| = |2||\vec{a}| = 2|\vec{a}| \neq |\vec{a}|$]

Remark. In this way, we can construct an infinite number of possible answers.

4. **Find the values of *x* and *y* so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.**

Sol. Given: $2\hat{i} + 3\hat{j} = x\hat{i} + y\hat{j}$.

Comparing coefficients of $\hat{i}$ and $\hat{j}$ on both sides, we have

$x = 2$ and $y = 3$.

5. **Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (– 5, 7).**

Sol. Let $\vec{AB}$ be the vector with initial point A(2, 1) and terminal point B(– 5, 7).

$\Rightarrow$ P.V. (Position Vector) of point A is (2, 1) = $2\hat{i} + \hat{j}$ and P.V. of point B is (– 5, 7) = $-5\hat{i} + 7\hat{j}$.

$\therefore\ \vec{AB}$ = P.V. of point B – P.V. of point A

$= (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j}) = -5\hat{i} + 7\hat{j} - 2\hat{i} - \hat{j}$

$\Rightarrow \vec{AB} = -7\hat{i} + 6\hat{j}$.

$\therefore$ By definition, scalar components of the vectors $\vec{AB}$ are coefficients of $\hat{i}$ and $\hat{j}$ in $\vec{AB}$ *i.e.,* – 7 and 6 and vector components of the vector $\vec{AB}$ are $-7\hat{i}$ and $6\hat{j}$.

6. Find the sum of the vectors:

$$\vec{a} = \hat{i} - 2\hat{j} + \hat{k},\ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$$

and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Sol. Given: $\vec{a} = \hat{i} - 2\hat{j} + \hat{k},\ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$

and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Adding $\vec{a} + \vec{b} + \vec{c} = 0\hat{i} - 4\hat{j} - \hat{k} = -4\hat{j} - \hat{k}$.

7. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Sol. We know that a unit vector in the direction of the vector

$$\vec{a} = \hat{i} + \hat{j} + 2\hat{k} \text{ is } \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{1+1+4}}$$

$$\Rightarrow \hat{a} = \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}.$$

8. Find the unit vector in the direction of the vector $\vec{PQ}$ **where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.**

Sol. Because points P and Q are P(1, 2, 3) and Q(4, 5, 6) (given),

therefore, position vector of point P = $\vec{OP} = 1\hat{i} + 2\hat{j} + 3\hat{k}$

and position vector of point Q = $\vec{OQ} = 4\hat{i} + 5\hat{j} + 6\hat{k}$

where O is the origin.

$\therefore$ PQ = Position vector of point Q – Position vector of point P

$$= \vec{OQ} - \vec{OP} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Therefore, a unit vector in the direction of vector $\vec{PQ}$

$$= \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{\sqrt{9+9+9=27=9\times 3}}$$

$$= \frac{3(\hat{i}+\hat{j}+\hat{k})}{3\sqrt{3}} = \frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}.$$

9. **For given vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$; find the unit vector in the direction of $\vec{a} + \vec{b}$.**

Sol. Given: Vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{j}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

$\therefore \quad \vec{a} + \vec{b} = 2\hat{i} - \hat{j} + 2\hat{k} - \hat{i} + \hat{j} - \hat{k} = \hat{i} + 0\hat{j} + \hat{k}$

$\therefore \quad |\vec{a} + \vec{b}| = \sqrt{(1)^2 + (0)^2 + (1)^2} = \sqrt{2}$

$\therefore$ A unit vector in the direction of $\vec{a} + \vec{b}$ is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{\hat{i}+0\hat{j}+\hat{k}}{\sqrt{2}} = \frac{\hat{i}+\hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}.$$

10. **Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.**

Sol. Let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$.

$\therefore$ A vector in the direction of vector $\vec{a}$ which has magnitude 8 units

$$= 8\hat{a} = 8\frac{\vec{a}}{|\vec{a}|} = \frac{8(5\hat{i}-\hat{j}+2\hat{k})}{\sqrt{25+1+4}}$$

$$= \frac{8}{\sqrt{30}}(5\hat{i} - \hat{j} + 2\hat{k}) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}.$$

11. **Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.**

Sol. Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ ...(*i*)

and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$

$= -2(2\hat{i} - 3\hat{j} + 4\hat{k}) = -2\vec{a}$ [By (*i*)]

$\Rightarrow \vec{b} = -2\vec{a} = m\vec{a}$ where $m = -2 < 0$

$\therefore$ Vectors $\vec{a}$ and $\vec{b}$ are collinear (unlike because $m = -2 < 0$).

12. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

Sol. The given vector is $(\vec{a}) = \hat{i} + 2\hat{j} + 3\hat{k}$

$$= \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

We know that direction cosines of a vector $\vec{a}$ are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{a}$ *i.e.*, $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

13. Find the direction cosines of the vector joining the points A(1, 2, – 3) and B(– 1, – 2, 1) directed from A to B.

Sol. Given: Points A(1, 2, – 3) and B(– 1, – 2, 1).

$$\underset{(1,\,2,\,-3)}{A} \longrightarrow \underset{(-1,\,-2,\,1)}{B}$$

$\Rightarrow$ P.V. (Position Vector, $\vec{OA}$) of point A is A(1, 2, – 3) $= \hat{i} + 2\hat{j} - 3\hat{k}$

and P.V. of point B is B(– 1, – 2, 1) $= -\hat{i} - 2\hat{j} + \hat{k}$.

$\therefore$ Vector $\vec{AB}$ (directed from A to B)

= P.V. of point B – P.V. of point A

$= -\hat{i} - 2\hat{j} + \hat{k} - (\hat{i} + 2\hat{j} - 3\hat{k})$

$= -\hat{i} - 2\hat{j} + \hat{k} - \hat{i} - 2\hat{j} + 3\hat{k} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

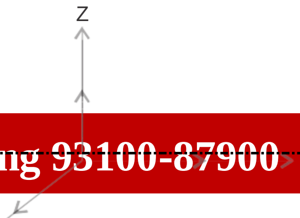
$\therefore$ AB $= |\vec{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4 + 16 + 16} = 6$

$\therefore$ A unit vector along $\vec{AB} = \frac{\vec{AB}}{|\vec{AB}|}$

$$= \frac{-2\hat{i} - 4\hat{j} + 4\hat{k}}{6} = -\frac{2}{6}\hat{i} - \frac{4}{6}\hat{j} + \frac{4}{6}\hat{k} = \frac{-1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}.$$

We know that Direction Cosines of the vector $\vec{AB}$ are the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in a unit vector along $\vec{AB}$ *i.e.*, $\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}$.

14. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

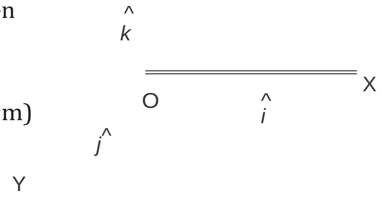


Sol. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Let us find angle θ_1 (say) between vector $\vec{a}$ and OX ($\Rightarrow \hat{i}$)

($\because$ $\hat{i}$ represents OX in vector form)

$\therefore \cos\theta_1 = \dfrac{\vec{a}\cdot\hat{i}}{|\vec{a}||\hat{i}|}$



$\Rightarrow \quad \cos\theta_1 = \dfrac{(\hat{i}+\hat{j}+\hat{k}).(\hat{i}+0\,\hat{j}+0\,\hat{k})}{|\hat{i}+\hat{j}+\hat{k}||\hat{i}+0\,\hat{j}+0\,\hat{k}|}$

$\Rightarrow \quad \cos\theta_1 = \dfrac{1(1)+1(0)+1(0)}{\sqrt{1+1+1}\,\sqrt{1+0+0}} = \dfrac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta_1 = \cos^{-1}\dfrac{1}{\sqrt{3}}$

Similarly, angle θ_2 between vectors $\vec{a}$ and $\hat{j}$ (OY) is $\cos^{-1}\dfrac{1}{\sqrt{3}}$

and angle θ_3 between vectors $\vec{a}$ and $\hat{k}$ (OZ) is also $\cos^{-1}\dfrac{1}{\sqrt{3}}$.

$\therefore \quad \theta_1 = \theta_2 = \theta_3$.

$\therefore$ Vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ is equally inclined to OX, OY and OZ

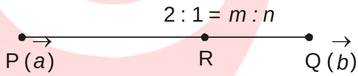
15. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1 (i) internally (*ii*) externally.

Sol. P.V. of point P is $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$

and P.V. of point Q is $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$

(i) Therefore P.V. of point R dividing PQ internally (*i.e.,* R lies within the segment PQ) in the ratio 2 : 1 (= *m* : *n*) (= PR : QR)

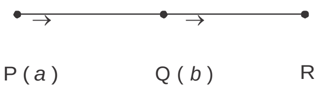
is $\dfrac{m\vec{b} + m\vec{a}}{m + m}$



$= \dfrac{2(-\hat{i}+\hat{j}+\hat{k})+\hat{i}+2\hat{j}-\hat{k}}{2+1} = \dfrac{-2\hat{i}+2\hat{j}+2\hat{k}+\hat{i}+2\hat{j}-\hat{k}}{3}$

$= \dfrac{-\hat{i}+4\hat{j}+\hat{k}}{3} = \dfrac{-1}{3}\hat{i} + \dfrac{4}{3}\hat{j} + \dfrac{1}{3}\hat{k}$.

(ii) P.V. of point R dividing PQ externally (*i.e.,* R lies outside PQ and to the right of point Q because ratio 2 : 1 = $\dfrac{2}{1}$ > 1 as PR is 2 times PQ *i.e.,* $\left(\dfrac{PR}{QR} = \dfrac{2}{1}\right)$ is $\dfrac{m\vec{b} - m\vec{a}}{m - m}$



$= \dfrac{2(-\hat{i}+\hat{j}+\hat{k})-(\hat{i}+2\hat{j}-\hat{k})}{2-1}$

$= -2\hat{i} + 2\hat{j} + 2\hat{k} - \hat{i} - 2\hat{j} + \hat{k} = -3\hat{i} + \hat{k}.$

Remark. In the above question 15(*ii*), had R been dividing PQ externally in the ratio 1 : 2; then R will lie to the left of point P and $\frac{PÆ}{QÆ} = \frac{1}{2}$.

16. Find the position vector of the mid-point of the vector joining the points P(2, 3, 4) and Q(4, 1, – 2).

Sol. Given: Point P is (2, 3, 4) and Q is (4, 1, – 2).

$\Rightarrow$ P.V. of point P(2, 3, 4) is $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

and P.V. of point Q(4, 1, – 2) is $\vec{b} = 4\hat{i} + \hat{j} - 2\hat{k}$.

$\therefore$ P.V. of mid-point R of PQ is $\dfrac{\vec{a}+\vec{b}}{2}$.

[By Formula of Internal division]

$$= \frac{2\hat{i}+3\hat{j}+4\hat{k}+4\hat{i}+\hat{j}-2\hat{k}}{2} = \frac{6\hat{i}+4\hat{j}+2\hat{k}}{2} = 3\hat{i} + 2\hat{j} + \hat{k}.$$

17. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right-angled triangle.

Sol. Given: P.V. of points A, B, C respectively are $\vec{a}\ (= \vec{OA}) = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b}\ (= \vec{OB}) = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c}\ (= \vec{OC}) = \hat{i} - 3\hat{j} - 5\hat{k}$, where O is the origin.

Step I. $\therefore$ $\vec{AB}$ = P.V. of point B – P.V. of point A

$= 2\hat{i} - \hat{j} + \hat{k} - (3\hat{i} - 4\hat{j} - 4\hat{k}) = 2\hat{i} - \hat{j} + \hat{k} - 3\hat{i} + 4\hat{j} + 4\hat{k}$

or $\vec{AB} = -\hat{i} + 3\hat{j} + 5\hat{k}$ ...(*i*)

$\vec{BC}$ = P.V. of point C – P.V. of point B

$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 5\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= -\hat{i} - 2\hat{j} - 6\hat{k}$ ...(*ii*)

$\vec{AC}$ = P.V. of point C – P.V. of point A

$= \hat{i} - 3\hat{j} - 5\hat{k} - (3\hat{i} - 4\hat{j} - 4\hat{k}) = \hat{i} - 3\hat{j} - 5\hat{k} - 3\hat{i} + 4\hat{j} + 4\hat{k}$

$= -2\hat{i} + \hat{j} - \hat{k}$ ...(*iii*)

Adding (i) and (ii),

$$\vec{AB} + \vec{BC} = -\hat{i} + 3\hat{j} + 5\hat{k} - \hat{i} - 2\hat{j} - 6\hat{k}$$

$$-2\hat{i} + \hat{j} - \hat{k} = \vec{AC} \qquad \text{[By (iii)]}$$

$\therefore$ By Triangle Law of addition of Vectors, Points A, B, C are the Vertices of a triangle or points A, B, C are collinear.

Step II.

From (i) AB $= |\vec{AB}| = \sqrt{1+9+25} = \sqrt{35}$

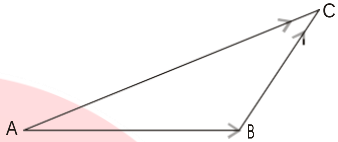
From (*ii*), BC = $|\overrightarrow{BC}| = \sqrt{1+4+36} = \sqrt{41}$

From (*iii*), AC = $|\overrightarrow{AC}| = \sqrt{4+1+1} = \sqrt{6}$

We can observe that (Longest side BC)^2 = $(\sqrt{41})^2 = 41 = 35 + 6$
$= AB^2 + AC^2$

$\therefore$ Points A, B, C are the vertices of a right-angled triangle.

18. In triangle ABC (Fig. below), which of the following is not true:



(A) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(B) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(C) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

(D) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

Sol. Option (C) is not true.

Because we know by Triangle Law of Addition of vectors that

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, *i.e.*, $\boxed{\begin{array}{l}\overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA} \\ \Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}\end{array}}$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

But for option (C), $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{AC} = 2\overrightarrow{AC} \neq \vec{0}$.

Option (D) is same as option (A).

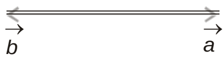
19. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

(A) $\vec{b} = \lambda\vec{a}$**, for some scalar λ.** **(B)** $\vec{a} = \pm\vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are proportional

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes.

Sol. Option (D) is not true because two collinear vectors can have **different** directions and also different magnitudes.



The options (A) and (C) are true by definition of collinear vectors.
Option (B) is a particular case of option (A) (taking $\lambda = \pm 1$).

Exercise 10.3

1. **Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 2, respectively having $\vec{a}\,.\,\vec{b} = \sqrt{6}$.**

Sol. Given: $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a}\,.\,\vec{b} = \sqrt{6}$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$. We know that

$\cos\theta$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Putting values, $\cos\theta = \dfrac{\sqrt{6}}{\sqrt{3}(2)}$

$= \dfrac{\sqrt{6}}{\sqrt{3}\sqrt{4}} = \dfrac{\sqrt{6}}{\sqrt{12}} = \sqrt{\dfrac{6}{12}} = \sqrt{\dfrac{1}{2}} = \dfrac{1}{\sqrt{2}} = \cos\dfrac{\pi}{4}$ $\quad\therefore\ \theta = \dfrac{\pi}{4}$.

2. Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

Sol. Given: Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

$\therefore |\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$ $\left[\because |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + x^2}\right.$

and $|\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$

Also, $\vec{a}.\vec{b}$ = Product of coefficients of $\hat{i}$ + Product of coefficient of $\hat{j}$ + Product of coefficients of $\hat{k}$

$= 1(3) + (-2)(-2) + 3(1) = 3 + 4 + 3 = 10$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

We know that $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{10}{\sqrt{14}\sqrt{14}} = \dfrac{10}{14} = \dfrac{5}{7}$

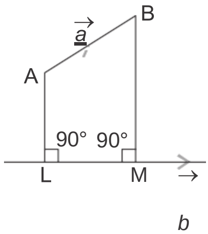
$\therefore\ \theta = \cos^{-1}\dfrac{5}{7}$.

3. Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

Sol. Let $\vec{a} = \hat{i} - \hat{j} = \hat{i} - \hat{j} + 0\hat{k}$

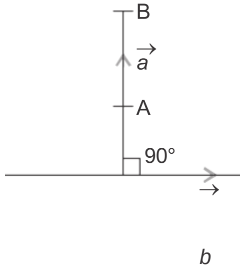
and $\vec{b} = \hat{i} + \hat{j} = \hat{i} + \hat{j} + 0\hat{k}$

Projection of vector $\vec{a}$ and $\vec{b}$



= Length LM = $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \dfrac{(1)(1)+(-1)(1)+0(0)}{\sqrt{(1)^2+(1)^2+0^2}} = \dfrac{0}{\sqrt{2}}$

$= \frac{1 - 1 + 0}{}$ $=$ $= 0.$



Remark. If projection of vector $\vec{a}$ on $\vec{b}$ is zero, then vector $\vec{a}$ is perpendicular to vector $\vec{b}$.

4. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Sol. Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

We know that projection of vector $\vec{a}$ on vector $\vec{b}$ = $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{1(7)+3(-1)+7(8)}{\sqrt{(7)^2+(-1)^2+(8)^2}} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}.$$

5. Show that each of the given three vectors is a unit vector: $\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$, $\frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$, $\frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$.

Also show that they are mutually perpendicular to each other.

Sol. Let

$$\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k} \quad ...(i)$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k} \quad ...(ii)$$

$$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}) = \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k} \quad ...(iii)$$

$$\therefore \quad |\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}}$$

$$= \sqrt{\frac{49}{49}} = \sqrt{1} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = \sqrt{\frac{49}{49}}$$

$$= \sqrt{1} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = \sqrt{\frac{49}{49}}$$

$$= \sqrt{1} = 1$$

$\therefore$ Each of the three given vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ From (i)

and (*ii*), is a unit vector.

$$\vec{a}\cdot\vec{b} = \left(\frac{2}{7}\right)\cdot\left(\frac{3}{7}\right) + \left(\frac{3}{7}\right)\left(\frac{-6}{7}\right) + \left(\frac{6}{7}\right)\left(\frac{2}{7}\right)$$

$$[\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3]$$

$$= \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = \frac{6-18+12}{49} = \frac{0}{49} = 0$$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

From (*ii*) and (*iii*),

$$\vec{b} \cdot \vec{c} = \left(\frac{3}{7}\right)\left(\frac{6}{7}\right) + \left(\frac{-6}{7}\right)\left(\frac{2}{7}\right) + \frac{2}{7}\left(\frac{-3}{7}\right)$$

$$= \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = \frac{18-12-6}{49} = \frac{0}{49} = 0$$

$\therefore$ $\vec{b}$ and $\vec{c}$ are perpendicular to each other.

From (*i*) and (*iii*),

$$\vec{a} \cdot \vec{c} = \frac{2}{7}\left(\frac{6}{7}\right) + \frac{3}{7}\left(\frac{2}{7}\right) + \left(\frac{6}{7}\right)\left(\frac{-3}{7}\right)$$

$$= \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = \frac{12+6-18}{49} = \frac{0}{49} = 0$$

$\therefore$ $\vec{a}$ and $\vec{c}$ are perpendicular to each other.

Hence, $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors.

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Sol. Given: $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$ ...(*i*)

$\Rightarrow$ $\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 8$

$\Rightarrow$ $|\vec{a}|^2 - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} - |\vec{b}|^2 = 8$

[$\because$ We know that $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ and $\vec{b} \cdot \vec{b} = |\vec{b}|^2$ and

$. \vec{a} = \vec{a} . \vec{b}$]

$\Rightarrow \quad |\vec{a}|^2 - |\vec{b}|^2 = 8$ ...(*ii*)

Putting $|\vec{a}| = 8|\vec{b}|$ from (*i*) in (*ii*), $64\,|\vec{b}|^2 - |\vec{b}|^2 = 8$

or $(64 - 1)\,|\vec{b}|^2 = 8 \quad \Rightarrow 63\,|\vec{b}|^2 = 8$

$\Rightarrow \quad |\vec{b}|^2 = \frac{8}{63} \quad \Rightarrow \quad |\vec{b}| = \sqrt{\frac{8}{63}} = \sqrt{\frac{4 \times 2}{9 \times 7}}$

(∵ Length *i.e.*, modulus of a vector is never negative.)

$\Rightarrow \quad |\vec{b}| = \frac{2}{3}\sqrt{\frac{2}{7}}$

Putting this value of $|\vec{b}|$ in (*i*),

$$|\vec{a}| = 8\left(\frac{2}{3}\sqrt{\frac{2}{7}}\right) = \frac{16}{3}\sqrt{\frac{2}{7}}.$$

7. Evaluate the product $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$.

Sol. The given expression $= (3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$

$= (3\vec{a}) \cdot (2\vec{a}) + (3\vec{a}) \cdot (7\vec{b}) - (5\vec{b}) \cdot (2\vec{a}) - (5\vec{b}) \cdot (7\vec{b})$

$= 6\,\vec{a} \cdot \vec{a} + 21\,\vec{a} \cdot \vec{b} - 10\,\vec{b} \cdot \vec{a} - 35\,\vec{b} \cdot \vec{b}$

$= 6|\vec{a}|^2 + 21\,\vec{a} \cdot \vec{b} - 10\,\vec{a} \cdot \vec{b} - 35|\vec{b}|^2$

$[\because \vec{a} \cdot \vec{a} = |\vec{a}|^2$ and $\vec{b} \cdot \vec{b} = |\vec{b}|^2$ and $\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}]$

$= 6|\vec{a}|^2 + 11\,\vec{a} \cdot \vec{b} - 35|\vec{b}|^2.$

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

Sol. Given: $|\vec{a}| = |\vec{b}|$ and angle θ (say) between $\vec{a}$ and $\vec{b}$ is 60° and their scalar (*i.e.,* dot) product $= \frac{1}{2}$

i.e., $\vec{a} \cdot \vec{b} = \frac{1}{2}$

$\Rightarrow |\vec{a}|\,|\vec{b}| \cos\theta = \frac{1}{2}$ $\quad [\because \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta]$

Putting $|\vec{b}| = |\vec{a}|$ (given) and θ = 60° (given), we have

$|\vec{a}|\,|\vec{a}| \cos 60° = \frac{1}{2} \quad \Rightarrow \quad |\vec{a}|^2 \left(\frac{1}{2}\right) = \frac{1}{2}$

Multiplying by 2, $|\vec{a}|^2 = 1 \quad \Rightarrow \quad |\vec{a}| = 1$ ...(*i*)

($\because$ Length of a vector is never negative)

$\therefore \quad |\vec{b}| = |\vec{a}| = 1$ [By (*i*)]

$\therefore \quad |\vec{a}| = 1$ and $|\vec{b}| = 1.$

9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$.

Sol. Given: $\vec{a}$ is a unit vector $\Rightarrow |\vec{a}| = 1$ ...(*i*)

Also given $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$

$$\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}.\vec{x} - \vec{a}.\vec{a} = 12$$

$$\Rightarrow |\vec{x}|^2 + \vec{a}.\vec{x} - \vec{a}.\vec{x} - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

Putting $|\vec{a}| = 1$ from (*i*), $|\vec{x}|^2 - 1 = 12$

$\Rightarrow \quad |\vec{x}|^2 = 13 \qquad\qquad \Rightarrow \quad |\vec{x}| = \sqrt{13}$.

($\because$ Length of a vector is never negative.)

10. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Sol. Given : $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$

and $\quad \vec{c} = 3\hat{i} + \hat{j}$.

Now, $\vec{a} + \lambda\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k} + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$

$= 2\hat{i} + 2\hat{j} + 3\hat{k} - \lambda\hat{i} + 2\lambda\hat{j} + \lambda\hat{k}$

$\Rightarrow \quad \vec{a} + \lambda\vec{b} = (2 - \lambda)\hat{j} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}$

Again given $\vec{c} = 3\hat{i} + \hat{j} = 3\hat{i} + \hat{j} + 0\hat{k}$.

Because vector $\vec{a} + \lambda\vec{a}$ is perpendicular to $\vec{c}$, therefore,

$(\vec{a} + \lambda\vec{b}) \cdot \vec{c} = 0$

i.e., Product of coefficients of $\hat{i}$ +............= 0

$\Rightarrow \quad (2 - \lambda)3 + (2 + 2\lambda)1 + (3 + \lambda)0 = 0$

$\Rightarrow \quad 6 - 3\lambda + 2 + 2\lambda = 0 \qquad \Rightarrow \quad -\lambda + 8 = 0$

$\Rightarrow \quad -\lambda = -8 \qquad \Rightarrow \quad \lambda = 8$.

11. Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a}$ and $\vec{b}$.

Sol. Let $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a} = l\vec{b} + m\vec{a}$

where $l = |\vec{a}|$ and $m = |\vec{b}|$

Let $\quad \vec{d} = |\vec{a}|\vec{b} - |\vec{b}|\vec{a} = l\vec{b} - m\vec{a}$

Now, $\vec{c}\,.\,\vec{d} = (l\vec{b} + m\vec{a})\,.\,(l\vec{b} - m\vec{a})$

$= l^2\,\vec{b}\,.\,\vec{b} - lm\,\vec{b}\,.\,\vec{a} + lm\,\vec{a}\,.\,\vec{b} - m^2\,\vec{a}\,.\,\vec{a}$

$= l^2|\vec{b}|^2 - lm\,\vec{a}\,.\,\vec{b} + lm\,\vec{a}\,.\,\vec{b} - m^2|\vec{a}|^2 = l^2|\vec{b}|^2 - m^2|\vec{a}|^2$

Putting $l = |\vec{a}|$ and $m = |\vec{b}|$,

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{b}|^2|\vec{a}|^2 = 0$

i.e., $\vec{c}\,.\,\vec{d} = 0$

$\therefore$ Vectors $\vec{c}$ and $\vec{d}$ are perpendicular to each other.

12. If $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

Sol. Given: $\vec{a}\cdot\vec{a} = 0 \Rightarrow |\vec{a}|^2 = 0 \Rightarrow |\vec{a}| = 0$ ...(*i*)

($\Rightarrow \vec{a}$ is a zero vector by definition of zero vector.)

Again given $\vec{a}\cdot\vec{b} = 0 \Rightarrow |\vec{a}||\vec{b}|\cos\theta = 0$

Putting $|\vec{a}| = 0$ from (*i*), we have $0|\vec{b}|\cos\theta = 0$

i.e., $0 = 0$ for all (any) vectors $\vec{b}$. $\therefore$ $\vec{b}$ can be any vector.

Note. $(\vec{a} + \vec{b} + \vec{c})^2 = (\vec{a} + (\vec{b} + \vec{c}))^2$

$= \vec{a}^2 + (\vec{b} + \vec{c})^2 + 2\vec{a}\cdot(\vec{b} + \vec{c})$ $[\because (\vec{A} + \vec{B})^2 = \vec{A}^2 + \vec{B}^2 + 2\vec{A}\cdot\vec{B}]$

$= \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2\vec{b}\cdot\vec{c} + 2\vec{a}\cdot\vec{b} + 2\vec{a}\cdot\vec{c}$

Using $\vec{a}\cdot\vec{c} = \vec{c}\cdot\vec{a}$

or $(\vec{a} + \vec{b} + \vec{c})^2 = \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$

13. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$.

Sol. Because $\vec{a}, \vec{b}, \vec{c}$ are unit vectors, therefore,

$|\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{c}| = 1$ ....(*i*)

Again given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Squaring both sides $(\vec{a} + \vec{b} + \vec{c})^2 = 0$

Using formula of **Note** above

$\Rightarrow \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

or $|\vec{a}| + |\vec{b}| + |\vec{c}| + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

Putting $|\vec{a}| = 1$, $|\vec{b}| = 1$, $|\vec{c}| = 1$ from (*i*),

$$1 + 1 + 1 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$$

$$\Rightarrow \quad 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = -3$$

Dividing both sides by 2, $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = \dfrac{-3}{2}$.

14. If either vector $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a}.\vec{b} = 0$. But the converse need not be true. Justify your answer with an example.

Sol. **Case I.** Vector $\vec{a} = \vec{0}$. Therefore, by definition of zero vector,

$|\vec{a}| = 0$ ...(*i*)

$\therefore \quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 0\,(|\vec{b}|\cos\theta)$ [By (*i*)]

$= 0$

Case II. Vector $\vec{b} = \vec{0}$. Proceeding as above we can prove that

$\vec{a}\cdot\vec{b} = 0$

But the converse is not true.
Let us justify it with an example.

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$. Therefore, $|\vec{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3} \neq 0$.

Therefore $\vec{a} \neq \vec{0}$ (By definition of Zero Vector)

Let $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$.

Therefore, $|\vec{b}| = \sqrt{(1)^2+(1)^2+(-2)^2} = \sqrt{6} \neq 0$.

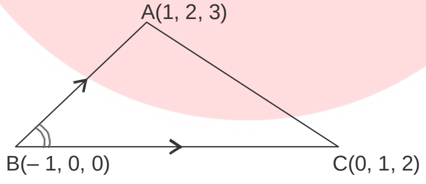
Therefore, $\vec{b} \neq \vec{0}$.

But $\vec{a}\cdot\vec{b} = 1(1) + 1(1) + 1(-2) = 1 + 1 - 2 = 0$

So here $\vec{a}\cdot\vec{b} = 0$ but neither $\vec{a} = \vec{0}$ nor $\vec{b} = \vec{0}$.

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (– 1, 0, 0) and (0, 1, 2), respectively, then find ∠ABC.

Sol. Given: Vertices A, B, C of a triangle are A(1, 2, 3), B(– 1, 0, 0) and C(0, 1, 2) respectively.



$\therefore$ Position vector (P.V.) of point A (=s $\vec{OA}$) = (1, 2, 3)

$= \hat{i} + 2\hat{j} + 3\hat{k}$

Position vector (P.V.) of point B (= $\vec{OB}$) = (– 1, 0, 0)

$= -\hat{i} + 0\hat{j} + 0\hat{k}$

and position vector (P.V.) of point C (= $\vec{OC}$) = (0, 1, 2)

$= 0\hat{i} + \hat{j} + 2\hat{k}$

We can see from the above figure that $\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$

Now $\overrightarrow{BA}$ = P.V. of terminal point A – P.V. of initial point B

$= \hat{i} + 2\hat{j} + 3\hat{k} - (-\hat{i} + 0\hat{j} + 0\hat{k})$

$= \hat{i} + 2\hat{j} + 3\hat{k} + \hat{i} - 0\hat{j} - 0\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ ...(*i*)

and $\overrightarrow{BC}$ = P.V. of point C – P.V. of point B

$= 0\hat{i} + \hat{j} + 2\hat{k} - (-\hat{i} + 0\hat{j} + 0\hat{k})$

$= 0\hat{i} + \hat{j} + 2\hat{k} + \hat{i} - 0\hat{j} - 0\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$ ...(*ii*)

We know that $\cos \angle ABC = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$ | $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

Using (*i*) and (*ii*)

$$= \frac{2(1)+2(1)+3(2)}{\sqrt{4+4+9}\,\sqrt{1+1+4}} = \frac{10}{\sqrt{17}\sqrt{6}} = \frac{10}{\sqrt{102}}$$

$\therefore \quad \angle ABC = \cos^{-1}\dfrac{10}{\sqrt{102}}$.

16. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, – 1) are collinear.

Sol. Given points are A(1, 2, 7), B(2, 6, 3) and C(3, 10, – 1).

$\Rightarrow$ P.V.'s $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ of points A, B, C are

$\overrightarrow{OA} = (1, 2, 7) = \hat{i} + 2\hat{j} + 7\hat{k}$

$\overrightarrow{OB} = (2, 6, 3) = 2\hat{i} + 6\hat{j} + 3\hat{k}$

and $\overrightarrow{OC} = (3, 10, -1) = 3\hat{i} + 10\hat{j} - \hat{k}$

$\therefore$ $\overrightarrow{AB}$ = P.V. of terminal point B – P.V. of initial point A

$= 2\hat{i} + 6\hat{j} + 3\hat{k} - (\hat{i} + 2\hat{j} + 7\hat{k})$

$= 2\hat{i} + 6\hat{j} + 3\hat{k} - \hat{i} - 2\hat{j} - 7\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$ ...(*i*)

and $\overrightarrow{AC}$ = P.V. of point C – P.V. of point A

$$= 3\hat{i} + 10\hat{j} - \hat{k} - (\hat{i} + 2\hat{j} + 7\hat{k})$$

$$= 3\hat{i} + 10\hat{j} - \hat{k} - \hat{i} - 2\hat{j} - 7\hat{k}$$

$$= 2\hat{i} + 8\hat{j} - 8\hat{k} = 2(\hat{i} + 4\hat{j} - 4\hat{k})$$

$\Rightarrow \quad \overrightarrow{AC} = 2\overrightarrow{AB}$ [By (*i*)]

$\Rightarrow$ Vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear or parallel. $|\because \vec{a} = m\vec{b}$
$\Rightarrow$ Points A, B, C are collinear.

($\because$ Vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have a common point A and hence can't be parallel.)

Remark. When we come to exercise 10.4 and learn that Exercise, we have a second solution for proving points A, B, C to be collinear: Prove that $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}$.

17. Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle.

Sol. Let the given (position) vectors be P.V.'s of the points A, B, C respectively.

P.V. of point A is $2\hat{i} - \hat{j} + \hat{k}$ and

P.V. of point B is $\hat{i} - 3\hat{j} - 5\hat{k}$ and

P.V. of point C is $3\hat{i} - 4\hat{j} - 4\hat{k}$.

$\therefore$ $\overrightarrow{AB}$ = P.V. of point B – P.V. of point A

$= \hat{i} - 3\hat{j} - 5\hat{k} - (2\hat{i} - \hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 5\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= -\hat{i} - 2\hat{j} - 6\hat{k}$ ...(*i*)

and $\overrightarrow{BC}$ = P.V. of point C – P.V. of point B

$= 3\hat{i} - 4\hat{j} - 4\hat{k} - (\hat{i} - 3\hat{j} - 5\hat{k}) = 3\hat{i} - 4\hat{j} - 4\hat{k} - \hat{i} + 3\hat{j} + 5\hat{k}$

$= 2\hat{i} - \hat{j} + \hat{k}$ ...(*ii*)

and $\overrightarrow{AC}$ = P.V. of point C – P.V. of point A

$= 3\hat{i} - 4\hat{j} - 4\hat{k} - (2\hat{i} - \hat{j} + \hat{k}) = 3\hat{i} - 4\hat{j} - 4\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= \hat{i} - 3\hat{j} - 5\hat{k}$ ...(*iii*)

Adding (i) and (ii), we have

$\overrightarrow{AB} + \overrightarrow{BC} = -\hat{i} - 2\hat{j} - 6\hat{k} + 2\hat{i} - \hat{j} + \hat{k}$

$= \hat{i} - 3\hat{j} - 5\hat{k} = \overrightarrow{AC}$ [By (iii)]

$\therefore$ By Triangle Law of addition of vectors, points A, B, C are the vertices of a triangle ABC or points A, B, C are collinear.

Now from (i) and (ii), $\overrightarrow{AB} . \overrightarrow{BC} = (-1)(2) + (-2)(-1) + (-6)(1)$

$= -2 + 2 - 6 = -6 \neq 0$

From (*ii*) and (*iii*), $\vec{BC}.\vec{AC} = 2(1) + (-1)(-3) + 1(-5)$
$= 2 + 3 - 5 = 0$

$\Rightarrow$ $\vec{BC}$ is perpendicular to $\vec{AC}$

$\Rightarrow$ Angle C is 90°. $\therefore$ ΔABC is right angled at point C.

$\therefore$ Points A, B, C are the vertices of a right angled triangle.

18. If $\vec{a}$ is a non-zero vector of magnitude '*a*' and λ is a non-zero scalar, then $\lambda\vec{a}$ is a unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = \frac{1}{|\lambda|}$

Sol. Given: $\vec{a}$ is a non-zero vector of magnitude *a*

$\Rightarrow |\vec{a}| = 1$ ...(*i*)

Also given: $\lambda \neq 0$ and $\lambda\vec{a}$ is a unit vector.

$\Rightarrow |\lambda\vec{a}| = 1$ $\Rightarrow |\lambda||\vec{a}| = 1$

$\Rightarrow |\lambda|\, a = 1$ $\Rightarrow$ $a = \frac{1}{|\lambda|}$

$\therefore$ Option (D) is the correct answer.

Exercise 10.4

1. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Sol. Given: $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Therefore, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$

$\left[\because \text{ If } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k};\right.$

$\left.\text{then } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\right]$

Expanding along first row,

$$\vec{a} \times \vec{b} = \hat{i}\begin{vmatrix} -7 & 7 \\ -2 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 7 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -7 \\ 3 & -2 \end{vmatrix}$$

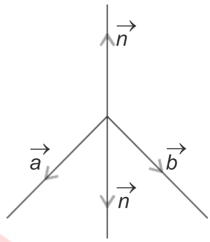
$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}(-14 + 14) - \hat{j}(2 - 21) + \hat{k}(-2 + 21)$$

$$= 0\hat{i} + 19\hat{j} + 19\hat{k}$$

$$\therefore \ |\vec{a} \times \vec{b}| = \sqrt{0^2 + (19)^2 + (19)^2} = \sqrt{2(19)^2} = \sqrt{2}\,(19) = 19\sqrt{2}.$$

Result: We know that $\vec{n} = \vec{a} \times \vec{b}$ is a vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.

Therefore, a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$ is



$$\hat{m} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}. \qquad \left[\because \hat{A} = \frac{\vec{A}}{|\vec{A}|}\right]$$

2. Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Sol. Given: $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Adding, $\vec{c} = \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j} + 0\hat{k}$

Subtracting $\vec{d} = \vec{a} - \vec{b} = 2\hat{i} + 0\hat{j} + 4\hat{k}$

Therefore, $\vec{n} = \vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

Expanding along first row $= \hat{i}(16 - 0) - \hat{j}(16 - 0) + \hat{k}(0 - 8)$

$$\Rightarrow \quad \vec{n} = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\therefore |\vec{n}| = \sqrt{(16)^2 + (-16)^2 + (-8)^2} = \sqrt{256 + 256 + 64} = \sqrt{576} = 24.$$

Therefore, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\hat{n} = \pm \frac{\vec{n}}{|\vec{n}|} = \pm \frac{(16\hat{i} - 16\hat{j} - 8\hat{k})}{24}$$

$$= \pm\left(\frac{16}{24}\hat{i} - \frac{16}{24}\hat{j} - \frac{8}{24}\hat{k}\right) = \pm\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right).$$

3. If a unit vector $\hat{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\hat{a}$.

Sol. Let $\hat{a} = x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector ...(*i*)

$\Rightarrow \quad |\hat{a}| = 1 \Rightarrow \sqrt{x^2 + y^2 + x^2} = 1$

Squaring both sides, $x^2 + y^2 + z^2 = 1$ ...(*ii*)

Given: Angle between vectors $\hat{a}$ and $\hat{i} = \hat{i} + 0\hat{j} + 0\hat{k}$ is $\frac{\pi}{3}$.

$\therefore \quad \cos\frac{\pi}{3} = \frac{\hat{a}.\hat{i}}{|\hat{a}||\hat{i}|} \qquad \left[\because \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$

$\Rightarrow \quad \frac{1}{2} = \frac{x(1)+y(0)+x(0)}{(1)(1)}$ or $\frac{1}{2} = x$ ...(*iii*)

Again, **Given:** Angle between vectors $\hat{a}$ and $\hat{j} = 0\hat{i} + \hat{j} + 0\hat{k}$ is $\frac{\pi}{4}$.

$\therefore \quad \cos\frac{\pi}{4} = \frac{\hat{a}.\hat{j}}{|\hat{a}||\hat{j}|} \Rightarrow \frac{1}{\sqrt{2}} = \frac{x(0)+y(1)+x(0)}{(1)(1)}$

$\Rightarrow \quad \frac{1}{\sqrt{2}} = y$ ...(*iv*)

Again, **Given:** Angle between vectors $\hat{a}$ and $\hat{k} = 0\hat{i} + 0\hat{j} + \hat{k}$ is θ where θ is acute.

$\therefore \quad \cos\theta = \frac{\hat{a}.\hat{k}}{|\hat{a}||\hat{k}|} = \frac{x(0)+y(0)+x(1)}{(1)(1)} = z$ ...(*v*)

Putting values of x, y and z from (*iii*), (*iv*) and (*v*) in (*ii*),

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$\Rightarrow \quad \cos^2\theta = 1 - \frac{1}{4} - \frac{1}{2} = \frac{4-1-2}{4} = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$

But θ is acute angle (given)

$\Rightarrow \quad \cos\theta$ is positive and hence $= \frac{1}{2} = \cos\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

From (v), $z = \cos\theta = \dfrac{1}{2}$

Putting values of x, y, z in (i), $\hat{a} = \dfrac{1}{2}\hat{i} + \dfrac{1}{\sqrt{2}}\hat{j} + \dfrac{1}{2}\hat{k}$

$\therefore$ Components of $\hat{a}$ are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{a}$

i.e., $\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{2}$ and acute angle $\theta = \dfrac{\pi}{3}$.

4. Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$.

Sol. L.H.S. $= (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$

$[\because \vec{a} \times \vec{a} = \vec{0},\ \vec{b} \times \vec{b} = \vec{0}$ and $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}]$

$= 2\vec{a} \times \vec{b}$ = R.H.S.

5. Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

Sol. Given: $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{0}$$

Expanding along first row,

$\hat{i}(6\mu - 27\lambda) - \hat{j}(2\mu - 27) + \hat{k}(2\lambda - 6) = \vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

Comparing coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides, we have

$6\mu - 27\lambda = 0$ ...(*i*)

$2\mu - 27 = 0$ ...(*ii*)

and $2\lambda - 6 = 0$ ...(*iii*)

From (*ii*), $2\mu = 27 \Rightarrow \mu = \frac{27}{2}$

From (*iii*), $2\lambda = 6 \Rightarrow \lambda = \frac{6}{2} = 3$

Putting $\lambda = 3$ and $\mu = \frac{27}{2}$ in (*i*), $6\left(\frac{27}{2}\right) - 27(3) = 0$

or $81 - 81 = 0$ or $0 = 0$ which is true. $\therefore$ $\lambda = 3$ and $\mu = \frac{27}{2}$.

6. Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you

conclude about the vectors $\vec{a}$ and $\vec{b}$?

Sol. Given: $\vec{a} \cdot \vec{b} = 0 \Rightarrow |\vec{a}|\,|\vec{b}| \cos\theta = 0$

$\Rightarrow$ Either $|\vec{a}| = 0$

or $|\vec{b}| = 0$ or $\cos\theta = 0$ $(\Rightarrow \theta = 90°)$ $\vec{a}$

$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

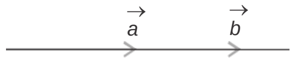
or vector $\vec{a}$ is perpendicular to $\vec{b}$..(*i*)

($\because$ By definition, vector $\vec{a}$ is zero vector if and only if $|\vec{a}| = 0$)

Again given $\vec{a} \times \vec{b} = \vec{0}$ $\Rightarrow |\vec{a} \times \vec{b}| = 0$

$\Rightarrow |\vec{a}|\,|\vec{b}| \sin\theta = 0$ $[\because |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta]$

$\Rightarrow$ Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\sin\theta = 0$ ($\Rightarrow$ $\theta = 0$)



$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or vectors $\vec{a}$ and $\vec{b}$ are collinear (or parallel) vectors. ....(*ii*)

We know from common sense that vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other as well as are parallel (or collinear) is impossible................(*iii*)

$\therefore$ From (*i*), (*ii*) and (*iii*), either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

$\therefore$ $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$

$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

7. Let the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Sol. Given: Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$\therefore$ $\vec{b} + \vec{c} = (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$

L.H.S. $= \vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

[By Property of Determinants]

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ = R.H.S.

8. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Sol. Given: Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

$\therefore\ |\vec{a}| = |\vec{0}| = 0$ or $|\vec{b}| = |\vec{0}| = 0$ ...(*i*)

$\therefore\ |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta = 0\ (\sin\theta) = 0$ [By (*i*)]

$\therefore\ \vec{a} \times \vec{b} = \vec{0}$ (By definition of zero vector)

But the converse is not true.

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ $\therefore\ |\vec{a}| = \sqrt{1+1+1} = \sqrt{3} \neq 0.$

$\therefore\ \vec{a}$ is a non-zero vector.

Let $|\vec{b}| = 2(\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore\ |\vec{b}| = \sqrt{4+4+4}$ or $|\vec{b}| = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3} \neq 0.$

$\therefore\ \vec{b}$ is a non-zero vector.

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix}$

Taking 2 common from R_3, $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \vec{0}$

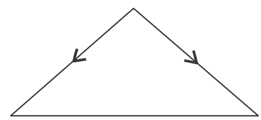
($\because$ R_2 and R_3 are identical)

9. **Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).**

Sol. Vertices of ΔABC are A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

$\therefore$ Position Vector (P.V.) of point A is (1, 1, 2) $= \hat{i} + \hat{j} + 2\hat{k}$

P.V. of point B is (2, 3, 5)



$= 2\hat{i} + 3\hat{j} + 5\hat{k}$

P.V. of point C is (1, 5, 5)

$= \hat{i} + 5\hat{j} + 5\hat{k}$

$\therefore\ \vec{AB}$ = P.V. of point B – P.V. of point A

= 2 $\hat{i}$ =

$\rightarrow$ i

$+ 3\hat{j} + 5\hat{k} - (\hat{i} + \hat{j} + 2\hat{k})$

$+ 3\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$

and AC = P.V. of point C − P.V. of point A

$= \hat{i} + 5\hat{j} + 5\hat{k} - (\hat{i} + \hat{j} + 2\hat{k}) = \hat{i} + 5\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k}$

$= 0\hat{i} + 4\hat{j} + 3\hat{k}$

$$\therefore \quad \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$$= \hat{i}(6 - 12) - \hat{j}(3 - 0) + \hat{k}(4 - 0) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

We know that **area of triangle ABC**

$$= \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{36 + 9 + 16} \quad | \quad \sqrt{x^2 + y^2 + x^2}$$

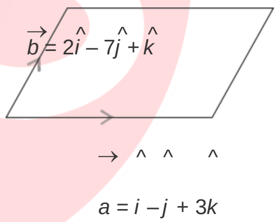
$$= \frac{1}{2}\sqrt{61} \text{ sq. units.}$$

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Sol. Given: Vectors representing two adjacent sides of a parallelogram are

$$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$$

and $$\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}.$$



$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$$= \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

We know that **area of parallelogram** $= |\vec{a} \times \vec{b}|$

$$= \sqrt{400 + 25 + 25} = \sqrt{450} = \sqrt{25 \times 9 \times 2}$$

$$= 5(3)\sqrt{2} = 15\sqrt{2} \text{ square units.}$$

Note. Area of parallelogram whose **diagonal vectors** are $\vec{\alpha}$ and $\vec{\beta}$ is $\frac{1}{2}|\vec{\alpha} \times \vec{\beta}|$.

11. **Let the vectors** $\vec{a}$ **and** $\vec{b}$ **be such that** $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$, **then** $\vec{a} \times \vec{b}$ **is a unit vector, if the angle between** $\vec{a}$ **and** $\vec{b}$ **is**

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$.

Sol. Given: $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector.

$\Rightarrow \ |\vec{a} \times \vec{b}| = 1 \qquad \Rightarrow \ |\vec{a}|\,|\vec{b}| \sin\theta = 1$

where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.

Putting values of $|\vec{a}|$ and $|\vec{b}|$, $3\left(\frac{\sqrt{2}}{3}\right)\sin\theta = 1$

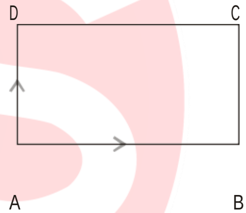
$\Rightarrow \ \sqrt{2}\sin\theta = 1 \ \Rightarrow \ \sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \ \Rightarrow \ \theta = \frac{\pi}{4}$

$\therefore$ Option (B) is the correct answer.

12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively, is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Sol. Given: ABCD is a rectangle.



We know that $\vec{AB}$ = P.V. of point B – P.V. of point A

$= \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k} - \left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$

$= \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k} + \hat{i} - \frac{1}{2}\hat{j} - 4\hat{k} = 2\hat{i} + 0\hat{j} + 0\hat{k}$

$\therefore \ AB = |\vec{AB}| = \sqrt{4+0+0} = \sqrt{4} = 2$

and $\vec{AD}$ = P.V. of point D – P.V. of point A

$= -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k} - \left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$

$= -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k} + \hat{i} - \frac{1}{2}\hat{j} - 4\hat{k} = -\hat{j} = 0\hat{i} - \hat{j} + 0\hat{k}$

$\therefore \ AD = |\vec{AD}| = \sqrt{0+1+0} = \sqrt{1} = 1$

$\therefore$ Area of rectangle ABCD = (AB)(AD) (= Length × Breadth)

= 2(1) = 2 sq. units

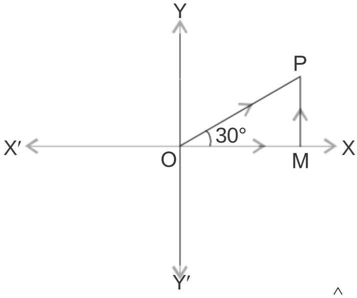
$\therefore$ Option (C) is the correct answer.

or Area of rectangle ABCD = $\left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$.

MISCELLANEOUS EXERCISE

1. **Write down a unit vector in XY-plane making an angle of 30° with the positive direction of *x*-axis.**

Sol. Let $\overrightarrow{OP}$ be the **unit** vector in XY-plane such that $\angle XOP = 30°$



Therefore, $|\overrightarrow{OP}| = 1$

i.e., $OP = 1$ ...(*i*)

By Triangle Law of Addition of vectors,

In ΔOMP, $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$

$= (OM)\hat{i} + (MP)\hat{j}$

$\left[\because \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} \Rightarrow \vec{a} = |\vec{a}|\,\hat{a}\right.$ and unit vector along OX is $\hat{i}$

and along OY is $\left.\hat{j}\right]$

$\Rightarrow \overrightarrow{OP} = OP\dfrac{OM}{OP}\hat{i} + OP\dfrac{MP}{OP}\hat{j}$

(Dividing and multiplying by OP in R.H.S.)

$= (1)(\cos 30°)\hat{i} + (1)(\sin 30°)\hat{j}$ $[\because$ By (*i*), $OP = 1]$

$\Rightarrow$ unit vector $\overrightarrow{OP} = (\cos 30)\,i + (\sin 30°)\,j$ ... (*ii*)

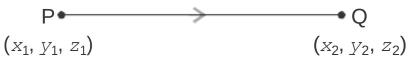
$\Rightarrow \overrightarrow{OP} = \dfrac{\sqrt{3}}{2}\hat{i} + \dfrac{1}{2}\hat{j}$.

Remark: From Eqn. (*ii*) of above solution, we can generalise the following result.

A unit vector along a line making an angle θ with positive x-axis is $(\cos\theta)\hat{i} + (\sin\theta)\hat{j}$

2. Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Sol. Given points are $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.



$\Rightarrow$ P.V. (Position vector) of point P is

$(x_1, y_1, z_1) = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and P.V. of point Q is $(x_2, y_2, z_2) = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\therefore$ Vector $\overrightarrow{PQ}$, the vector joining the points P and Q.

= P.V. of terminal point Q − P.V. of initial point P

$$= x_2\hat{i} + y_2\hat{j} + z_2\hat{k} - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= x_2\hat{i} + y_2\hat{j} + z_2\hat{k} - x_1\hat{i} - y_1\hat{j} - z_1\hat{k}$$

$$\Rightarrow \overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$\therefore$ Scalar components of the vector $\overrightarrow{PQ}$ are the coefficients of

$\hat{i}, \hat{j}, \hat{k}$ in $\overrightarrow{PQ}$ *i.e.,* $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$

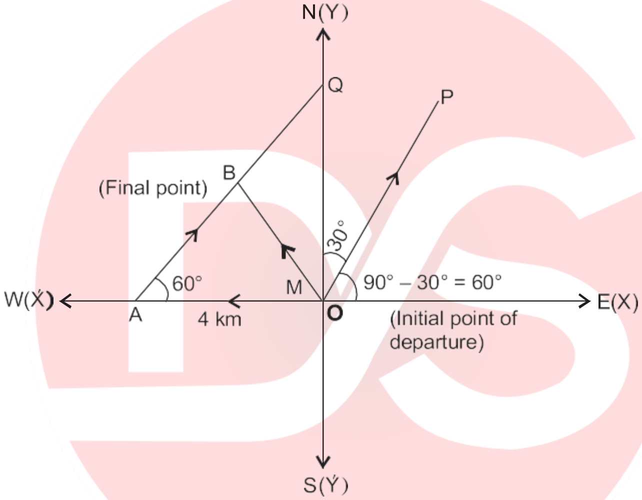
and magnitude of vector $\overrightarrow{PQ}$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (x_2 - x_1)^2} \;.\; | \;\sqrt{x^2 + y^2 + x^2}$$

3. **A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.**

Sol. Let us take the initial point of departure as origin.

Let the girl walk a distance OA = 4 km towards west.



Through the point A draw a line AQ parallel to a line OP (which is 30° east of North *i.e.,* in East-North quadrant making an angle of 30° with North)

Let the girl walk a distance AB = 3 km (given) along this direction $\overrightarrow{OQ}$ (given). $\therefore \overrightarrow{OA} = 4(-\hat{i})$[$\because$ Vector $\overrightarrow{OA}$ is along OX′)]

$$= -4\hat{i} \qquad ...(i)$$

We know that (By Remark Q.N. 1 of this miscellaneous exercise) a unit vector along $\overrightarrow{AQ}$ (or $\overrightarrow{AB}$) making an angle θ = 60° with positive *x*-axis is $(\cos\theta)\hat{i} + (\sin\theta)\hat{j} = (\cos 60°)\hat{i} + (\sin 60°)\hat{j}$

$$= \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}.$$

$\therefore \overrightarrow{AB} = |\overrightarrow{AB}|$ (A unit vector along $\overrightarrow{AB}$) $|\because \vec{a} = |\vec{a}|\hat{a}$

$$= 3\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) = \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j} \quad \ldots (ii)$$

$\therefore$ Girl's displacement from her initial point O of departure (to final point B) $= \overrightarrow{OB}$ of $= \overrightarrow{OA} + \overrightarrow{AB}$ (By Triangle Law of Addition vectors)

$$= -4\hat{i} + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right) = \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

[By (*i*)] [By (*ii*)]

$$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}.$$

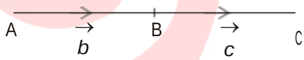
4. **If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.**

Sol. The result is not true (always).

Given: $\vec{a} = \vec{b} + \vec{c}$.

$\therefore$ Either the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are collinear or vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form the sides of a triangle.

Case I. Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are collinear.

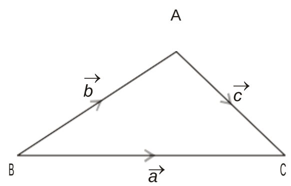


Let $\vec{a} = \overrightarrow{AC}$, $\vec{b} = \overrightarrow{AB}$ and $\vec{c} = \overrightarrow{BC}$,

then $\vec{a} = \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{b} + \vec{c}$.

Also, $|\vec{a}|$ = AC = AB + BC = $|\vec{b}| + |\vec{c}|$.

Case II. Vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form a triangle.



Here also by Triangle Law of vectors,

$$\vec{a} = \vec{b} + \vec{c}$$

But $|\vec{a}| < |\vec{b}| + |\vec{c}|$

($\because$ Each side of a triangle is less than sum of the other two sides)

$\therefore$ $|(\vec{a}) = \vec{b} + \vec{c}| = |\vec{b}| + |\vec{c}|$ is true only when vectors

$\vec{b}$ and $\vec{c}$ are collinear vectors.

5. Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Sol. Because $x(\hat{i} + \hat{j} + \hat{k}) = x\hat{i} + x\hat{j} + x\hat{k}$ is a unit vector (given)

Therefore, $|x\hat{i} + x\hat{j} + x\hat{k}| = 1$

$\therefore \quad \sqrt{x^2 + x^2 + x^2} = 1$

$\left[\because \; x\hat{i} + y\hat{j} + z\hat{k} = \sqrt{x^2 + y^2 + x^2}\,\right]$

Squaring both sides $3x^2 = 1$ or $x^2 = \frac{1}{3}$ $\quad \therefore \quad x = \pm \frac{1}{\sqrt{3}}$.

6. Find a vector of magnitude 5 units and parallel to the resultant of the vectors

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \text{ and } \vec{b} = \hat{i} - 2\hat{j} + \hat{k}.$$

Sol. Given: Vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Let vector $\vec{c}$ be the resultant of vectors $\vec{a}$ and $\vec{b}$.

$$\therefore \quad \vec{c} = \vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} - \hat{k} + \hat{i} - 2\hat{j} + \hat{k}.$$

$$= 3\hat{i} + \hat{j} + 0\hat{k}.$$

$\therefore$ Required vector of magnitude 5 units and parallel (or collinear) to resultant vector $\vec{c} = \vec{a} + \vec{b}$ is

$$5\hat{c} = 5\frac{\vec{c}}{|\vec{c}|} = 5\left(\frac{3\hat{i} + \hat{j} + 0\hat{k}}{\sqrt{9 + 1 + 0}}\right)$$

$$= \frac{5}{\sqrt{10}}(3\hat{i} + \hat{j}) = \frac{5}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}(3\hat{i} + \hat{j})$$

$$= \frac{5}{10}\sqrt{10}(3\hat{i} + \hat{j}) = \frac{\sqrt{10}}{2}(3\hat{i} + \hat{j}) = \frac{3}{2}\sqrt{10}\,\hat{i} + \frac{\sqrt{10}}{2}\hat{j}.$$

7. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Sol. Given: Vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$

and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$.

Let $\vec{d} = 2\vec{a} - \vec{b} + 3\vec{c}$

$= 2(\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k})$

$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 2\hat{i} + \hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}$

$\therefore \ \hat{d} = 3\hat{i} - 3\hat{j} + 2\hat{k}$ $\therefore$ A unit vector parallel to the vector $\vec{d} = 3\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{3\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{9+9+4} = \sqrt{22}} = \frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}.$$

8. Show that the points A(1, – 2, – 8), B(5, 0, – 2) and C(11, 3, 7) are collinear and find the ratio in which B divides AC.

Sol. Given: Points A(1, – 2, – 8), B(5, 0, – 2) and C(11, 3, 7).

i.e., Position vectors of points A, B, C are

$\vec{OA}$ (= A(1, – 2, – 8)) = $\hat{i} - 2\hat{j} - 8\hat{k}$

$\vec{OB}$ (= B(5, 0, – 2)) = $5\hat{i} + 0\hat{j} - 2\hat{k} = 5\hat{i} - 2\hat{k}$

and $\vec{OC}$ (= C(11, 3, 7)) = $11\hat{i} + 3\hat{j} + 7\hat{k}$

$\therefore$ $\vec{AB}$ = P.V. of point B – P.V. of point A

$= 5\hat{i} - 2\hat{k} - (\hat{i} - 2\hat{j} - 8\hat{k}) = 5\hat{i} - 2\hat{k} - \hat{i} + 2\hat{j} + 8\hat{k}$

or $\vec{AB} = 4\hat{i} + 2\hat{j} + 6\hat{k}$

$\therefore$ AB = $|\vec{AB}| = \sqrt{16+4+36} = \sqrt{56} = \sqrt{4 \times 14} = 2\sqrt{14}$

and $\vec{BC}$ = P.V. of point C – P.V. of point B

$= 11\hat{i} + 3\hat{j} + 7\hat{k} - (5\hat{i} - 2\hat{k}) = 11\hat{i} + 3\hat{j} + 7\hat{k} - 5\hat{i} + 2\hat{k}$

$= 6\hat{i} + 3\hat{j} + 9\hat{k}$

$\therefore$ BC = $|\vec{BC}| = \sqrt{36+9+81} = \sqrt{126} = \sqrt{9 \times 14} = 3\sqrt{14}$

$\vec{AC}$ = P.V. of point C – P.V. of point A

$= 11\hat{i} + 3\hat{j} + 7\hat{k} - (\hat{i} - 2\hat{j} - 8\hat{k})$

$= 11\hat{i} + 3\hat{j} + 7\hat{k} - \hat{i} + 2\hat{j} + 8\hat{k} = 10\hat{i} + 5\hat{j} + 15\hat{k}$

$\therefore$ AC = $|\vec{AC}| = \sqrt{100+25+225} = \sqrt{350} = \sqrt{25 \times 14} = 5\sqrt{14}$

Now, $\vec{AB} + \vec{BC} = 4\hat{i} + 2\hat{j} + 6\hat{k} + 6\hat{i} + 3\hat{j} + 9\hat{k}$

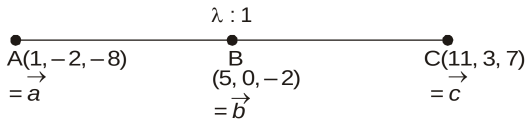
$= 10\hat{i} + 5\hat{j} + 15\hat{k} = \vec{AC}$

$\therefore$ Points A, B, C are either collinear or are the vertices of ΔABC.

Again AB + BC = $2\sqrt{14} + 3\sqrt{14} = (2+3)\sqrt{14} = 5\sqrt{14}$ = AC

$\therefore$ Points A, B, C are collinear.

Now to find the ratio in which B divides AC



Let the point B divides AC in the ratio $\lambda : 1$.

$\therefore$ By section formula, P.V. of point B is $\dfrac{\lambda\vec{c} + 1\vec{a}}{\lambda + 1}$

$\Rightarrow$ $(5, 0, -2) = \dfrac{\lambda(11, 3, 7) + (1, -2, -8)}{\lambda + 1}$

Cross-multiplying,

$$(\lambda + 1)(5\hat{i} + 0\hat{j} - 2\hat{k}) = \lambda(11\hat{i} + 3\hat{j} + 7\hat{k}) + (\hat{i} - 2\hat{j} - 8\hat{k})$$

$$\Rightarrow 5(\lambda + 1)\hat{i} - 2(\lambda + 1)\hat{k} = 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k}$$

$$\Rightarrow (5\lambda + 5)\hat{i} - (2\lambda + 2)\hat{k} = (11\lambda + 1)\hat{i} + (3\lambda - 2)\hat{j} + (7\lambda - 8)\hat{k}$$

Comparing coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ on both sides, we have

$5\lambda + 5 = 11\lambda + 1,\ 0 = 3\lambda - 2,\ -(2\lambda + 2) = 7\lambda - 8$

$\Rightarrow -6\lambda = -4,\ -3\lambda = -2,\ -2\lambda - 2 = 7\lambda - 8\ (\Rightarrow\ -9\lambda = -6)$

$\Rightarrow$ $\lambda = \dfrac{4}{6} = \dfrac{2}{3},\ \lambda = \dfrac{2}{3},\ \lambda = \dfrac{6}{9} = \dfrac{2}{3}$

All three values of λ are same.

$\therefore$ Required ratio is $\lambda : 1 = \dfrac{2}{3} : 1 = 2 : 3.$

9. Find the position vector of a point R which divides the line joining the two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1 : 2. Also, show that P is the middle point of line segment RQ.

Sol. We know that position vector of the point R dividing the join of P and Q externally in the ratio $1 : 2 = m : n$ is given by

$$\vec{c} = \frac{m\vec{b} - m\vec{a}}{m - m} = \frac{1(\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{1 - 2}$$

$$= \frac{\vec{a} - 3\vec{b} - 4\vec{a} - 2\vec{b}}{1 - 2} = \frac{-3\vec{a} - 5\vec{b}}{-1} = 3\vec{a} + 5\vec{b}$$

Again position vector of the middle point of the line segment RQ

$$= \frac{\text{P.V. of point Æ + P.V. of point Q}}{2} = \frac{3\vec{a} + 5\vec{b} + \vec{a} - 3\vec{b}}{2} = \frac{4\vec{a} + 2\vec{b}}{2}$$

$= 2\vec{a} + \vec{b}$ = P.V. of point P. (given)

$\therefore$ Point P is the middle point of the line segment RQ.

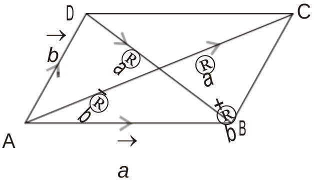
10. Two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Sol. Let ABCD be a parallelogram.

Given: The vectors representing two adjacent sides of this parallelogram are say

$\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$

and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$



Formula: $\therefore$ Vectors along the diagonals $\vec{AC}$ and $\vec{DB}$ of the parallelogram are

$$\vec{a} + \vec{b} \text{ and } \vec{a} - \vec{b}$$

i.e., $\vec{a} + \vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k} + \hat{i} - 2\hat{j} - 3\hat{k}$

$= 3\hat{i} - 6\hat{j} + 2\hat{k}$

and $\vec{a} - \vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k} - (\hat{i} - 2\hat{j} - 3\hat{k})$

$= 2\hat{i} - 4\hat{j} + 5\hat{k} - \hat{i} + 2\hat{j} + 3\hat{k} = \hat{i} - 2\hat{j} + 8\hat{k}$

$\therefore$ Unit vectors parallel to (or along) diagonals are

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} \text{ and } \frac{\vec{a}-\vec{b}}{|\vec{a}-\vec{b}|} = \frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4} = \sqrt{49} = 7} \text{ and } \frac{\hat{i}-2\hat{j}+8\hat{k}}{\sqrt{1+4+64} = \sqrt{69}}$$

Let us find area of parallelogram

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} = \hat{i}(12 + 10) - \hat{j}(-6 - 5) + \hat{k}(-4 + 4)$$

$= 22\hat{i} + 11\hat{j} + 0\hat{k}$

We know that area of parallelogram $= |\vec{a} \times \vec{b}|$

$= \sqrt{(22)^2 + (11)^2 + 0^2} = \sqrt{484 + 121} = \sqrt{605}$

$= \sqrt{5 \times 121} = \sqrt{121 \times 5} = 11\sqrt{5}$ sq. units.

11. **Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.**

Sol. Let l, m, n be the direction cosines of a vector equally inclined to the axes OX, OY, OZ.

$\therefore$ A unit vector along the given vector is

$\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$ and $|\hat{a}| = 1$

$\Rightarrow \sqrt{l^2 + m^2 + n^2} = 1 \qquad \therefore \quad l^2 + m^2 + n^2 = 1 \qquad \ldots(i)$

Let the given vector (for which unit vector is $\hat{a}$) make equal angles (given) θ, θ, θ (say) with **OX** ($\Rightarrow \hat{i}$), **OY** ($\Rightarrow \hat{j}$) and **OZ** ($\Rightarrow \hat{k}$)
$\therefore$ The given vector is in positive octant OXYZ and hence θ is acute..(*ii*)

$\therefore$ For angle θ between $\hat{a}$ and $\hat{i}$,

$$\cos\theta = \frac{\hat{a}\cdot\hat{i}}{|\hat{a}||\hat{i}|} = \frac{(l\hat{i}+m\hat{j}+n\hat{k})\cdot(\hat{i}+0\hat{j}+0\hat{k})}{(1)(1)}$$

or $\cos\theta = l(1) + m(0) + n(0) = l$

or $l = \cos\theta$ ...(*iii*)

Similarly, for angle θ between $\hat{a}$ and $\hat{j}$, $m = \cos\theta$ ...(*iv*)

Similarly, for angle θ between $\hat{a}$ and $\hat{k}$, $n = \cos\theta$ ...(*v*)

Putting these values of l, m, n from (*iii*), (*iv*) and (*v*) in (*i*), we have

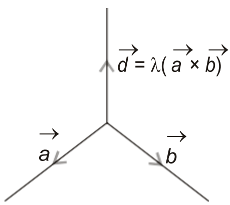
$$\cos^2\theta + \cos^2\theta + \cos^2\theta = 1 \quad\Rightarrow\quad 3\cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = \frac{1}{3} \quad\Rightarrow\quad \cos\theta = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$$

$\therefore$ $\cos\theta = \frac{1}{\sqrt{3}}$ ($\because$ By (*ii*), θ is acute and hence $\cos\theta$ is positive)

Putting $\cos\theta = \frac{1}{\sqrt{3}}$ in (*ii*), (*iii*) and (*iv*), direction cosines of the required vector are $l, m, n = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

12. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}\cdot\vec{d} = 15$.



Sol. Given: Vectors are $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$

and $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$

By definition of cross-product of two vectors, $\vec{a}\times\vec{b}$ is *a* vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Hence, vector $\vec{d}$ which is also perpendicular to both $\vec{a}$ and $\vec{b}$

is $\vec{d} = \lambda(\vec{a} \times \vec{b})$ where $\lambda = 1$ or some other scalar.

Therefore, $\vec{d} = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$

Expanding along first row, $= \lambda[\hat{i}(28 + 4) - \hat{j}(7 - 6) + \hat{k}(-2 - 12)]$

or $\vec{d} = \lambda[32\hat{i} - \hat{j} - 14\hat{k}]$ ...(*i*) or $\vec{d} = 32\lambda\hat{i} - \lambda\hat{j} - 14\lambda\hat{k}$

To find λ: Given: $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

Also given $\vec{c} \cdot \vec{d} = 15$

$\Rightarrow 2(32\lambda) + (-1)(-\lambda) + 4(-14\lambda) = 15$

$\Rightarrow 64\lambda + \lambda - 56\lambda = 15 \Rightarrow 9\lambda = 15 \Rightarrow \lambda = \frac{5}{3}$

Putting $\lambda = \frac{5}{3}$ in (*i*), required vector

$$\vec{d} = \frac{5}{3}(32\hat{i} - \hat{j} - 14\hat{k}) = \frac{1}{3}(160\hat{i} - 5\hat{j} - 70\hat{k}).$$

13. **The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.**

Sol. Given: Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ...(*i*)

$\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore \vec{b} + \vec{c}\ (= \vec{d}$ (say)$) = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

$\therefore \hat{d}$, a unit vector along $\vec{b} + \vec{c} = \vec{d}$ is

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 36 + 4} = \sqrt{4 + \lambda^2 + 4\lambda + 40}}$$

or $$\hat{d} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

$$= \frac{(2+\lambda)}{\sqrt{\lambda^2 + 4\lambda + 44}}\hat{i} + \frac{6}{\sqrt{\lambda^2 + 4\lambda + 44}}\hat{j} - \frac{2}{\sqrt{\lambda^2 + 4\lambda + 44}}\hat{k}$$ ...(*ii*)

Given: Scalar (*i.e.,* Dot) Product of $\vec{a}$ and $\hat{d}$ *i.e.,* $= \vec{a} \cdot \hat{d} = 1$

$\therefore$ From (*i*) and (*ii*),

$$\frac{1(2+\lambda)}{\sqrt{\lambda^2 + 4\lambda + 44}} + \frac{1(6)}{\sqrt{\lambda^2 + 4\lambda + 44}} + \frac{1(-2)}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

Multiplying by L.C.M. = $\sqrt{\lambda^2 + 4\lambda + 44}$,

$2 + \lambda + 6 - 2 = \sqrt{\lambda^2 + 4\lambda + 44}$ $\Rightarrow \lambda + 6 = \sqrt{\lambda^2 + 4\lambda + 44}$

Squaring both sides $(\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$

$\Rightarrow \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

$\Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1.$

14. **If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors of equal magnitude, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$, $\vec{c}$.**

Sol. Given: $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitude.

$\therefore \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = 0,\ \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{b} = 0,\ \vec{a}\cdot\vec{a} = \vec{a}\cdot\vec{c} = 0$ ... (*i*)

and $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say) ... (*ii*)

Let vector $\vec{d} = \vec{a} + \vec{b} + \vec{c}$ make angles $\theta_1, \theta_2, \theta_3$ with vectors $\vec{a}, \vec{b}, \vec{c}$ respectively.

$$\therefore \cos\theta_1 = \frac{\vec{d}\cdot\vec{a}}{|\vec{d}||\vec{a}|} = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} \quad \text{[By } (i)]$$

$$\Rightarrow \cos\theta_1 = \frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} \quad \text{... } (iii)$$

Let us now find $|\vec{a}+\vec{b}+\vec{c}|$.

We know that $|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})^2$

$= \vec{a}^{\,2} + (\vec{b}+\vec{c})^2 + 2\vec{a}\cdot(\vec{b}+\vec{c})$

$[\because (\vec{A}+\vec{B})^2 = \vec{A}^{\,2} + \vec{B}^{\,2} + 2\vec{A}\cdot\vec{B}]$

$= \vec{a}^{\,2} + \vec{b}^{\,2} + \vec{c}^{\,2} + 2\vec{b}\cdot\vec{c} + 2\vec{a}\cdot\vec{b} + 2\vec{a}\cdot\vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} + 2\vec{a}\cdot\vec{b} + 2\vec{a}\cdot\vec{c}$

Putting values from (*i*) and (*ii*)

$$|\vec{a}+\vec{b}+\vec{c}|^2 = \lambda^2 + \lambda^2 + \lambda^2 + 0 + 0 + 0 = 3\lambda^2$$

$$\therefore |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3\lambda^2} = \lambda\sqrt{3}$$

Putting this value of $|\vec{a}+\vec{b}+\vec{c}| = \lambda\sqrt{3}$ and $|\vec{a}| = \lambda$ from (*ii*) in (*iii*), $\cos\theta_1 = \frac{\lambda}{\lambda\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\quad \therefore \theta_1 = \cos^{-1}\frac{1}{\sqrt{3}}$

Similarly, $\theta_2 = \cos^{-1}\frac{1}{\sqrt{3}}$ and $\theta_3 = \cos^{-1}\frac{1}{\sqrt{3}}$

$\therefore \quad \theta_1 = \theta_2 = \theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore$ Vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

15. Prove that $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$, if and only if $\vec{a}$, $\vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$.

Sol. We know that $(\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + \vec{a}.\vec{b} + \vec{a}.\vec{b} + |\vec{b}|^2$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} \qquad ...(i)$$

For If part: Given: $\vec{a}$ and $\vec{b}$ are perpendicular

$\Rightarrow \vec{a}.\vec{b} = 0$

Putting $\vec{a}.\vec{b} = 0$ in (*i*), we have

$$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$$

For Only if part:

Given: $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$

Putting this value in L.H.S. eqn. (*i*), we have

$$|\vec{a}|^2 + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$$

$\Rightarrow 0 = 2\vec{a}.\vec{b}$ $\Rightarrow \vec{a}.\vec{b} = \frac{0}{2} = 0$

But $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$ (given).

$\therefore$ Vector $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

16. Choose the correct answer:
If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$ **(B)** $0 \leq \theta \leq \frac{\pi}{2}$ **(C)** $0 < \theta < \pi$ **(D)** $0 < \theta \leq \pi$

Sol. Given: $\vec{a}.\vec{b} \geq 0$

$\Rightarrow |\vec{a}||\vec{b}| \cos\theta \geq 0$ $\Rightarrow \cos\theta \geq 0$

[$\because$ $|\vec{a}|$ and $|\vec{b}|$ being lengths of vectors are always ≥ 0]

and this is true only for option (B) out of the given options $\left(\because \text{For option (A) } 0 < \theta < \frac{\pi}{2}, \cos\theta > 0\right)$.

17. Choose the correct answer :

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ **(B)** $\theta = \frac{\pi}{3}$ **(C)** $\theta = \frac{\pi}{2}$ **(D)** $\theta = \frac{2\pi}{3}$.

Sol. Given: $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$ are unit vectors

$\Rightarrow |\vec{a}| = 1$, $|\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = 1$

Now, squaring both sides of $|\vec{a} + \vec{b}| = 1$, we have

$$|\vec{a} + \vec{b}|^2 = 1 \quad \Rightarrow \quad (\vec{a} + \vec{b})^2 = 1$$

$$\Rightarrow \quad \vec{a}^{\,2} + \vec{b}^{\,2} + 2\,\vec{a}\cdot\vec{b} = 1$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}|\,|\vec{b}|\cos\theta = 1$$

where θ is the given angle between vectors $\vec{a}$ and $\vec{b}$.

Putting $|\vec{a}| = 1$ and $|\vec{b}| = 1$, we have $1 + 1 + 2\cos\theta = 1$

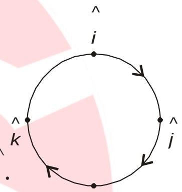
$$\Rightarrow \quad 2\cos\theta = -1 \qquad \Rightarrow \quad \cos\theta = \frac{-1}{2} = -\cos 60^\circ$$

$$\Rightarrow \quad \cos\theta = \cos(180^\circ - 60^\circ) \qquad \Rightarrow \quad \cos\theta = \cos 120^\circ$$

$$\Rightarrow \quad \theta = 120^\circ = 120 \times \frac{\pi}{180} = \frac{2\pi}{3}$$

$\therefore$ Option (D) is the correct answer.

Very Important Results



(1) $\hat{i}\cdot\hat{i} = |\hat{i}|^2 = 1,\ \hat{j}\cdot\hat{j} = 1,\ \hat{k}\cdot\hat{k} = 1.$

(2) $\hat{i} \times \hat{i} = \vec{0}\,,\ \hat{j} \times \hat{j} = \vec{0}$ and $\hat{k} \times \hat{k} = \vec{0}\,.$

(3) $\hat{i}\cdot\hat{j} = 0 = \hat{j}\cdot\hat{i},\ \hat{j}\cdot\hat{k} = 0 = \hat{k}\cdot\hat{j},\ \hat{i}\cdot\hat{k} = 0 = \hat{k}\cdot\hat{i}.$

(4) $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}.$

18. Choose the correct answer:

The value of $\hat{i}\cdot(\hat{j} \times \hat{k}) + \hat{j}\cdot(\hat{i} \times \hat{k}) + \hat{k}\cdot(\hat{i} \times \hat{j})$ is

(A) 0 **(B) − 1** **(C) 1** **(D) 3**

Sol. $\hat{i}\cdot(\hat{j} \times \hat{k}) + \hat{j}\cdot(\hat{i} \times \hat{k}) + \hat{k}\cdot(\hat{i} \times \hat{j})$

$$= \hat{i}\cdot\hat{i} + \hat{j}\cdot(-\hat{j}) + \hat{k}\cdot\hat{k}$$

$(\because \hat{i} \times \hat{k} = -\hat{k} \times \hat{i} = -\hat{j})$

$$= 1 - 1 + 1 = 1$$

$\therefore$ Option (C) is the correct answer.

19. If θ be the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}| = |\vec{a} \times \vec{b}|$, when θ is equal to

(A) 0 **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{2}$ **(D)** π

Sol. Given: $|\vec{a}\cdot\vec{b}| = |\vec{a} \times \vec{b}|$

$$\Rightarrow \quad |\vec{a}|\,|\vec{b}|\,|\cos\theta| = |\vec{a}|\,|\vec{b}|\sin\theta$$

$(\because \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow |\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}|\,|\cos\theta|)$

Dividing both sides by $|\vec{a}||\vec{b}|$, we have $|\cos\theta| = \sin\theta$

and this equation is true only for option (B) namely $\theta = \frac{\pi}{4}$ out of the given options.

$\left[\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and also } \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$

$\therefore$ Option (B) is the correct option.