Exercise 1.1

1. **Determine whether each of the following relations are reflexive, symmetric and transitive:**

   (i) Relation R in the set A = {1, 2, 3, ..., 13, 14} defined as

   $$R = \{(x, y) : 3x - y = 0\}$$

   (ii) Relation R in the set N of natural numbers defined as

   $$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$

   (iii) Relation R in the set A = {1, 2, 3, 4, 5, 6} as

   $$R = \{(x, y) : y \text{ is divisible by } x\}$$

   (iv) Relation R in the set Z of all integers defined as

   $$R = \{(x, y) : x - y \text{ is an integer}\}$$ **(*Important*)**

   (v) Relation R in the set A of human beings in a town at a particular time given by

   (*a*) R = {(x, y) : x and y work at the same place}

   (*b*) R = {(x, y) : x and y live in the same locality}

   (*c*) R = {(x, y) : x is exactly 7 cm taller than y}

   (*d*) R = {(x, y) : x is wife of y}

   (*e*) R = {(x, y) : x is father of y}

Sol. (*i*) **Given:** Set A = {1, 2, 3, ..., 13, 14}

Relation R in the set A is defined as

$R = \{(x, y) : 3x - y = 0\}$

i.e., $R = \{(x, y) : -y = -3x$ *i.e.*, $y = 3x\}$ ...(*i*)

Is R reflexive? Let $x \in A$. **Putting $y = x$** in (*i*), $x = 3x$

Dividing by $x \neq 0$ ($x \in A$ here is $\neq 0$), $1 = 3$ which is impossible.

Therefore $(x, x) \notin R$ and hence R is not reflexive. **Is R symmetric?** Let $(x, y) \in R$, therefore by (*i*) $y = 3x$

Interchanging x and y in (*i*), $x = 3y$ which is not true

$(\because\ y = 3x \Rightarrow x = \frac{y}{3}$ and $\neq 3y)$

For example, $2 \in A$, $6 \in A$ and $6 = 3(2)$ (*i.e.*, $y = 3x$) but $2 \neq 3(6) = 18$.

$\therefore$ $(2, 6) \in R$ but $(6, 2) \notin R$ $\therefore$ R is not symmetric.

Is R transitive? Let $(x, y) \in R$ and $(y, z) \in R$.

Therefore by (*i*), $y = 3x$ and $z = 3y$.

To eliminate y, Putting $y = 3x$ in $z = 3y$, $z = 3(3x) = 9x$

$\therefore$ By (*i*), $(x, z) \notin R$.

$\therefore$ R is not transitive.

(ii) **Given:** Relation R in the set of natural numbers defined as

$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$

i.e., $R = \{(x, x + 5) \text{ and } x = 1, 2, 3\}$

$\therefore$ In Roster form; $R = \{(1, 1 + 5), (2, 2 + 5), (3, 3 + 5)\}$

i.e., $R = \{(1, 6), (2, 7), (3, 8)\}$

Domain of R is the set of x co-ordinates of R. *i.e.,* the set $\{1, 2, 3\}$.

R is not reflexive because $(1, 1) \notin R$, $(2, 2) \notin R$, $(3, 3) \notin R$

R is not symmetric because $(1, 6) \in R$ but $(6, 1) \notin R$.

R is transitive because there are no two pairs of the type (x, y) and $(y, z) \in R = \{(1, 6), (2, 7), (3, 8)\}$;

so we should have no reason to expect $(x, z) \in R$.

Remark 1. Please note carefully and learn that relation R in the above question is transitive.

Remark 2. Whenever set A is a small finite set, it is always better to write R in roster form.

(iii) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ defined as $R = \{(x, y) : y \text{ is divisible by } x\}$.

$\therefore$ In roster form; $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

R is reflexive because $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)$ all $\in R$, *i.e.,* $(x, x) \in R$ for all $x \in A$.

R is not symmetric because $(2, 4) \in R$ but $(4, 2) \notin R$ as 2 is not divisible by 4.

Is R transitive? Let $x, y, z \in A$

such that $(x, y) \in R$ and $(y, z) \in R$.

$\therefore$ By definition of R in this question, y is divisible by x and z is divisible by y.

$\therefore$ There exist natural numbers m and n such that

$$y = mx \text{ and } z = ny.$$

To eliminate *y*: Putting $y = mx$ in $z = ny$, we have

$$z = n(mx) = (nm)x$$

$\therefore$ z is a multiple of, x, *i.e.,* z is divisible by x.

$\therefore$ $(x, z) \in R$. $\therefore$ R is transitive.

(iv) **Given:** Relation R in the set Z of all integers defined as

$R = \{(x, y) : (x - y) \text{ is an integer}\}$ ...(*i*)

Is R reflexive? Let $x \in Z$. Putting $y = x$ in (*i*),

$x - x = 0$ is an integer which is true.

$\therefore$ $(x, x) \in R$ for all $x \in Z$ $\therefore$ R is reflexive.

Is R symmetric? Let $x \in Z$, $y \in Z$, and $(x, y) \in R$.

$\therefore$ By (*i*) $(x - y)$ is an integer.

i.e., $-(y - x)$ is an integer and hence $y - x$ is an integer.

$\therefore$ $(y, x) \in R$. $\therefore$ R is symmetric.

Is R transitive? Let $x \in Z$, $y \in Z$ and $z \in Z$

such that $(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow$ By (*i*), $x - y$ is an integer and $y - z$ is also an integer.

Adding to eliminate y, $x - y + y - z$ = Integer + Integer

i.e., $x - z$ is an integer.

$\Rightarrow$ $(x, z) \in R$ by (*i*). $\therefore$ R is transitive.

(v) R is a relation in the set A of human beings of a town.

(a) **Given:** R = {(x, y) : x and y work at the same place} ...(i)

Is R reflexive? Let $x \in$ A.

Putting $y = x$ in (i), x and x work at the same place which is true ($\because$ x and x is just one person x)

Is R symmetric? Let $x \in$ A, $y \in$ A and (x, y) $\in$ R.

$\therefore$ By (i), x and y work at the same place.

i.e., same thing as y and x work at the same place. Therefore by (i), (y, x) $\in$ R.

Is R transitive? Let $x \in$ A, $y \in$ A, $z \in$ A such that (x, y) $\in$ R and (y, z) $\in$ R.

$\therefore$ By (i), x and y work at the same place.

Also By (i), y and z work at the same place.

Therefore, we can say that x and z also work at the same place.

$\therefore$ By (i), (x, z) $\in$ R. Therefore, R is transitive.

(b) **Same solution as of part (*a*)**

(Replace the word "work" by "live" and "place" by "locality" in the solution of part (a)).

(c) R = {(x, y) : x is exactly 7 cm taller than y) ...(i)

Is R reflexive? Let $x \in$ A. Putting $y = x$ in (i), we have x is exactly 7 cm taller than x, which is false.

($\because$ No body can be taller than oneself.)

Is R symmetric? Let $x \in$ A, $y \in$ A and (x, y) $\in$ R.

Therefore, by (i), x is exactly 7 cm taller than y.

$\therefore$ y is exactly 7 cm shorter than x.

$\therefore$ By (i), (y, x) $\notin$ R.

$\therefore$ R is not symmetric.

Is R transitive? Let $x \in$ A, $y \in$ A, $z \in$ A such that (x, y) $\in$ R and (y, z) $\in$ R.

$\therefore$ By (i), x is exactly 7 cm taller than y and y is exactly 7 cm taller than z.

$\therefore$ x is exactly (7 + 7) = 14 cm (and not 7 cm) taller than z.

$\therefore$ By (i), (x, z) $\notin$ R. $\therefore$ R is not transitive.

(d) **Given:** R = {(x, y) : x is wife of y} ...(i)

Is R reflexive? Let $x \in$ A. Putting $y = x$ in (i), we have x is wife of x which is false.

($\because$ No lady can be wife of herself.)

$\therefore$ R is not reflexive.

Is R symmetric? Let $x \in$ A, $y \in$ A and (x, y) $\in$ R.

$\therefore$ By (i), x is wife of y.

Hence y is husband (and not wife) of x.

$\therefore$ (y, x) $\notin$ R. $\therefore$ R is not symmetric.

Is R transitive?

There can't be any three $x, y, z \in$ A

(Set of Human beings) such that both $(x, y) \in$ R and $(y, z) \in$ R.

($\because$ $(x, y) \in$ R $\Rightarrow$ x is wife of y ($\Rightarrow$ Husband *i.e.,* Man) and hence (y, z) will never belong to R as no man y can be wife of any human being z).

Hence we no reason to expect that $(x, z) \in$ R. $\therefore$ R is transitive.

(e) **Given:** R = $\{(x, y) : x$ is father of $y\}$ ...(i)

Is R reflexive? Let $x \in$ A. Putting $y = x$ in (i), we have x is father of x which is false.

($\because$ No body can be father of oneself)

Is R symmetric? Let $x \in$ A, $y \in$ A such that $(x, y) \in$ R.

$\therefore$ By (i), x is father of y.

Hence y is son or daughter (and not father) of x.

$\therefore$ $(y, x) \notin$ R. $\therefore$ R is not symmetric.

Is R transitive? Let $x, y, z \in$ A such that $(x, y) \in$ R and $(y, z) \in$ R.

$\therefore$ By (i), x is father of y and y is father of z.

Hence x is grandfather (and not father) of z.

$\therefore$ $(x, z) \notin$ R. $\therefore$ R is not transitive.

2. Show that the relation R in the set R of real numbers, defined as R = $\{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Sol. Relation R = $\{(a, b) : a, b$ are real and $a \leq b^2\}$ ...(i)

Is R reflexive? Let a be any real number.

Putting $b = a$ in (i), $a \leq a^2$ which is not true for any positive real number less than 1.

For example, for $a = \frac{1}{2}$, $\frac{1}{2} \leq \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ (*i.e.,* $a \leq a^2$) is not true as $\frac{1}{2} > \frac{1}{4}$. $\therefore$ R is not reflexive.

Is R symmetric?

Let us take $a = 1$ and $b = 2$. Now $a = 1 \le 2^2 = 4$ $(= b^2)$ is true.

$\therefore$ By (*i*), $(a, b) \in$ R.1414

But $b = 2 > 1^2 = 1$ (a^2) *i.e.,* b is not less than equal to a^2, therefore $(b, a) \notin$ R. $\therefore$ R is not symmetric. **Is R transitive?**

Let us take $a = 10$, $b = 4$, and $c = 2$ (All three are real numbers)

Now by (*i*), $(a, b) = (10, 4) \in$ R ($\because$ $a = 10 \le b^2$ $(= 4^2)$ is true)

Again by (*i*), $(b, c) = (4, 2) \in$ R ($\because$ $b = 4 \le c^2$ $(= 2^2)$ is true)

($4 \le 2^2 \Rightarrow 4 \le 4 \Rightarrow$ Either $4 < 4$ or $4 = 4$ But $4 = 4$ is true).

But $(a, c) = (10, 2) \notin$ R [$\because$ $a = 10 > 2^2$ $(= b^2)$]

$\therefore$ R is not transitive.

$\therefore$ R is neither reflexive, nor symmetric nor transitive.

Remark. It may be noted that $4 \le 4$ is true. Also $4 \ge 4$ is true. $8 \ge 5$ is true and $3 \le 6$ is true.

3. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(*a*, *b*) : *b* = *a* + 1} is reflexive, symmetric or transitive.

Sol. Given: Set {1, 2, 3, 4, 5, 6} = set A (say)

Relation R $= \{(a, b) : b = a + 1\} = \{(a, a + 1) : a \in A\}$ ...(*i*)

Putting $a = 1, 2, 3, 4, 5, 6$ (given) in (*i*).

Roster form of relation R is {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)} ...(*ii*)

R is not reflexive. [$\because$ By (*ii*), $(1, 1) \notin$ R, $(2, 2) \notin$ R]

($\because$ We know that for relation R to be reflexive, $(a, a) \in$ R for all $a \in$ A)

R is not symmetric because by (*ii*), $(1, 2) \in$ R but $(2, 1) \notin$ R.

R is not transitive because $(x, y) = (1, 2) \in$ R, $(y, z) = (2, 3) \in$ R but $(x, z) = (1, 3) \notin$ R.

4. Show that the relation R in R defined as R = {(*a*, *b*) : *a* ≤ *b*}, is reflexive and transitive but not symmetric.

Sol. Given: Relation R in the set of real numbers is defined as

R $= \{(a, b) : a \le b\}$ ...(*i*)

Is R reflexive? Putting $b = a$ in (*i*), we have $a \le a$ which is true.

[$\because$ $a \le a \Rightarrow$ Either $a < a$ or $a = a$ and out of the two $a = a$ is true)

R is not **symmetric** because by (*i*), $(1, 2) \in$ R as $a = 1 \le b$ $(= 2)$ but $(2, 1) \notin$ R because b $(= 2) > 1(= a)$ (*i.e.,* $b \le a$ is not true).

Is R transitive? Let a, b, c be three real numbers such that $(a, b) \in$ R and $(b, c) \in$ R. $\therefore$ By (*i*), $a \le b$ and $b \le c$.

Therefore $a \le c$ and hence by (*i*), $(a, c) \in$ R. $\therefore$ R is transitive.

5. Check whether the relation R in R defined by R = {(*a*, *b*) : *a* ≤ *b*³} is reflexive, symmetric or transitive.

Sol. Relation R $= \{(a, b) : a, b$ are real and $a \le b^3\}$ ...(*i*)

Is R reflexive? Let a be any real number.

Putting $b = a$ in (i), $a \le a^3$ which is not true for any positive real number less than 1.

For example, for $a = \frac{1}{2}$, $\frac{1}{2} \le \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ (*i.e.*, $a \le a^3$) is not true

as $\frac{1}{2} > \frac{1}{8}$. $\therefore$ R is not reflexive.

Is R symmetric?

Let us take $a = 1$ and $b = 2$. Now $a = 1 \le 2^3 = 8$ $(= b^3)$ is true. Therefore by (i), $(a, b) \le$ R.

Now, $b = 2 > 1^3$ $(= 1)$ (a^3) *i.e.*, b is not less than or equal to a^3. Therefore, $(b, a) \notin$ R. $\therefore$ R is not symmetric.

Is R transitive?

Let us take $a = 10$, $b = 4$, $c = 2$ (All three are real numbers)

Now by (i), $(a, b) = (10, 4) \in$ R $(\because a = 10 \le b^3$ $(= 4^3 = 64$ is true)

Again by (i), $(b, c) = (4, 2) \in$ R $(\because b = 4 \le c^3$ $(= 2^3 = 8)$ is true.

But $(a, c) = (10, 2) \notin$ R $(\because a = 10 > 2^3$ $(= c^3 = 8))$.

$\therefore$ R is not transitive.

$\therefore$ R is neither reflexive nor symmetric nor transitive.

6. Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

Sol. Given: Set is $\{1, 2, 3\} = A$ (say)

Given relation $R = \{(1, 2), (2, 1)\}$ ...(*i*)

R is symmetric because $(x, y) = (1, 2) \in R$ and also $(y, x) = (2, 1) \in R$ and these two are the only elements of R.

i.e., $(x, y) \in R \Rightarrow (y, x) \in R$ for all $(x, y) \in R$.

R is not reflexive because $1 \in A$ but $(1, 1) \notin R$.

$2 \in A$ but $(2, 2) \notin R$, $3 \in A$ but $(3, 3) \notin R$.

R is not transitive because $(x, y) = (1, 2) \in R$ and $(y, z) = (2, 1) \in R$ but $(x, z) = (1, 1) \notin R$.

7. Show that the relation R in the set A of all the books in a library of a college given by R = {(*x*, *y*) : *x* and *y* have same number of pages} is an equivalence relation.

Sol. Given: Set A of all the books in a library of a college.

Given: Relation R : $\{(x, y)$: x and y have same number of pages$\}$ ...(*i*)

Is R reflexive? Let (book) $x \in$ A

Putting $y = x$ in (*i*), we have (book) x and x have same number of pages which is clearly true, because we are talking of pages of one book only.

$\therefore$ $(x, x) \in R$ for all $x \in A$. $\therefore$ R is reflexive.

Is R symmetric? Let $x \in A$, $y \in A$ and $(x, y) \in R$.

$\therefore$ By (*i*), books x and y have the same number of pages *i.e.,* books y and x have the same number of pages. Therefore $(y, x) \in R$.

$\therefore$ R is symmetric.

Is R transitive? Let $x, y, z \in A$ such that $(x, y) \in R$ and $(y, z) \in R$.

By (*i*), $(x, y) \in R \Rightarrow$ Books x and y have the same number of pages. .... (*ii*)

By (*i*), $(y, z) \in R \Rightarrow$ Books y and z have the same number of pages. .... (*iii*)

From (*ii*) and (*iii*) books x and z have the same number of pages. Therefore by (*i*), $(x, z) \in R$. $\therefore$ R is transitive. So have proved that the relation R given by (*i*) is reflexive, symmetric and transitive and hence is an equivalence relation.

8. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by R = {(*a*, *b*) : $|a - b|$ is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

Sol. (*i*) For all $a \in A$, $|a - a| = 0$ is even so that $(a, a) \in R$. Therefore, R is reflexive.

(ii) Further $(a, b) \in R \Rightarrow |a - b|$ is even
$\Rightarrow |b - a|$ is even, since $|b - a| = |-(a - b)|$
$= |a - b|$ $(\because |-t| = |t|)$
$\Rightarrow (b, a) \in R$ $\therefore$ R is symmetric.

(iii) Also $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow |a - b|$ is even and $|b - c|$ is even
$\Rightarrow a - b$ is even and $b - c$ is even
Adding the two $\Rightarrow a - c =$ even
[$\because$ addition of even numbers is even]
$\Rightarrow |a - c|$ is even $\Rightarrow (a, c) \in R$
$\therefore$ R is transitive.

Hence, R is an equivalence relation.

Further all the elements of {1, 3, 5} are related to each other since all the elements of this subset of A are odd and difference of two odd numbers is even.
[$\because |1 - 1| = 0, |3 - 3| = 0, |5 - 5| = 0$ are also even].

Similarly, all the elements of {2, 4} are related to each other since all the elements of this subset of A are even and difference of two even numbers is even.

Also, no element of {1, 3, 5} is related to any element of {2, 4} since the difference **of an odd number and an even number** is never even.
[$\because |1 - 2| = 1, |1 - 4| = 3, |3 - 2| = 1, |3 - 4| = 1,$
$|5 - 2| = 3, |5 - 4| = 1$ are odd and not even.]

9. Show that each of the relation R in the set $A = \{x \in Z, 0 \le x \le 12\}$ given by
(i) $R = \{(a, b) : |a - b|$ is a multiple of 4}
(ii) $R = \{(a, b) : a = b\}$
is an equivalence relation. Find the set of all elements related to 1 in each case.

Sol. (*i*) **Given:** Set $A = \{x \in Z, 0 \le x \le 12\}$
and $R = \{(a, b) : |a - b|$ is a multiple of 4} ...(*i*)

Is R reflexive? Let $a \in A$.
Putting $b = a$ (*i*), $|a - a| = |0| = 0$ is a multiple of 4.
which is true.
$\therefore (a, a) \in R$ for all $a \in A$. $\therefore$ R is reflexive.

Is R symmetric? Let $a \in A$, $b \in A$ and $(a, b) \in R$.
$\therefore$ By (*i*), $|a - b|$ is a multiple of 4.
i.e., $|-(b - a)| = |b - a|$
$(\because |-t| = |t|$ for every real $t)$ is a multiple of 4.
$\therefore$ By (*i*), $(b, a) \in R$. $\therefore$ R is symmetric. **Is R Transitive?** Let $a, b, c \in A$ such that $(a, b) \in R$ and $(b, c) \in R$.
By (*i*), $(a, b) \in R \Rightarrow |a - b|$ is a multiple of 4.
$\therefore$ There exists an integer m such that $|a - b| = 4m$

$\therefore \quad a - b = \pm 4m$ ...(*ii*)

Similarly, there exists an integer n such that $b - c = \pm 4n$ ...(*iii*)

Adding (*ii*) and (*iii*) (to eliminate b)

$$a - c = \pm 4m \pm 4n = \pm 4(m + n)$$

$\therefore \quad |a - c| = 4(m + n)$

i.e., $|a - c|$ is a multiple of 4.

$\therefore$ By (*i*), $(a, c) \in$ R. $\therefore$ R is transitive.

Hence R is an equivalence relation.

To find the set of elements related to 1 ($\in$ A).

Let $a \in$ A be related to $1 \in$ A.

$\therefore \quad (a, 1) \in$ R.

$\therefore$ By (*i*), $|a - 1|$ is a multiple of 4.

Testing a = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 $\in$ A, only a = 1, 5, 9 satisfy the above relation, *i.e.*, $|1 - 1| = 0$, $|5 - 1| = 4$, $|9 - 1| = 8$ are multiples of 4.

$\therefore$ The set of elements of A related to $1 \in$ A is {1, 5, 9}.

(*ii*) R = $\{(a, b) : a = b\}$ ...(*i*)

Is R reflexive? Putting $b = a$ in (*i*), we have $a = a$ which is true.

$\therefore \quad (a, a) \in$ R for all $a \in$ A. $\therefore$ R is reflexive.

Is R symmetric? Let $a \in$ A, $b \in$ A and $(a, b) \in$ R.

$\therefore$ By (*i*), $a = b$. Hence $b = a$.

$\therefore$ By (*i*), $(b, a) \in$ R. $\therefore$ R is symmetric.

Is R transitive? Let $a, b, c \in$ A such that

$(a, b) \in$ R and $(b, c) \in$ R.

$\therefore$ By (*i*), $a = b$ and $b = c$.

Hence $a = c$. Therefore by (*i*), $(a, c) \in$ R. $\therefore$ R is transitive.

Hence R is an equivalence relation.

To find set of elements of A related to 1 $\in$ A.

Let $a \in$ A be related to $1 \in$ A.

$\therefore \quad (a, 1) \in$ R. $\therefore$ By (*i*), $a = 1$.

$\therefore$ The set of elements of A related to $1 \in$ A is {1}.

10. Give an example of a relation, which is

(i) Symmetric but neither reflexive nor transitive.

(ii) Transitive but neither reflexive nor symmetric.

(iii) Reflexive and symmetric but not transitive.

(iv) Reflexive and transitive but not symmetric.

(v) Symmetric and transitive but not reflexive.

Sol. Let A = {1, 2, 3, 4}, then

A × A = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}

We know that a relation in A is a subset of A × A.

(i) The relation R = $\{(1, 2), (2, 1), (2, 4), (4, 2)\}$ is symmetric but neither reflexive nor transitive, because

$R^{-1} = \{(2, 1), (1, 2), (4, 2), (2, 4)\} = R$

$\Rightarrow$ R is symmetric

$1 \in A$ but $(1, 1) \notin R$ $\Rightarrow$ R is not reflexive

$(1, 2) \in R$ and $(2, 4) \in R$ but $(1, 4) \notin R$

$\Rightarrow$ R is not transitive.

(ii) The relation R = $\{(1, 2), (2, 3), (1, 3)\}$ is transitive but neither reflexive nor symmetric, because

$(1, 2) \in R$, $(2, 3) \in R$ and $(1, 3) \in R$

$\Rightarrow$ R is transitive

$1 \in A$ but $(1, 1) \notin R$ $\Rightarrow$ R is not reflexive

$(1, 2) \in R$ but $(2, 1) \notin R$ $\Rightarrow$ R is not symmetric.

(iii) The relation R = $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (1, 3), (3, 1)\}$ is reflexive and symmetric but not transitive, because

$(a, a) \in R$ for all $a \in A$ $\Rightarrow$ R is reflexive

$(a, b) \in R \Rightarrow (b, a) \in R$ $\Rightarrow$ R is symmetric

$(2, 1) \in R$ and $(1, 3) \in R$ but $(2, 3) \notin R$

$\Rightarrow$ R is not transitive.

(iv) The relation R = $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3)\}$ is reflexive and transitive but not symmetric, because

$(a, a) \in R$ for all $a \in A$ $\Rightarrow$ R is reflexive

If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

$\Rightarrow$ R is transitive

$(1, 2) \in R$ but $(2, 1) \notin R$ $\Rightarrow$ R is not symmetric

[Or $R^{-1} \neq R$ because $R^{-1} = \{(1, 1), (2, 2),$

(3, 3), (4, 4), (2, 1), (3, 2), (3, 1)} $\neq$ R.

$\Rightarrow$ R is not symmetric].

(v) The relation R = {(1, 1), (1, 2), (2, 1), (2, 2)} is symmetric and transitive but not reflexive, because

If $(a, b) \in R$ then $(b, a) \in R$ $\Rightarrow$ R is symmetric

[Or $R^{-1} = R$ $\Rightarrow$ R is symmetric]

If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

$\Rightarrow$ R is transitive

$3 \in A$ but $(3, 3) \notin R$ $\Rightarrow$ R is not reflexive.

11. Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P $\neq$ (0, 0) is the circle passing through P with origin as centre.

Sol. Given: Set A of points in a plane and relation R = {(P, Q) : distance of point P from origin is same as the distance of the point Q from the origin}.

i.e., R = {(P, Q) : OP = OQ where O is the origin} ...(*i*)

Is R reflexive? Let point $P \in A$.

Putting Q = P in (*i*), we have OP = OP which is true.

$\therefore$ $(P, P) \in R$ for all points $P \in A$. $\therefore$ R is reflexive.

Is R symmetric? Let points $P, Q \in A$ such that $(P, Q) \in R$.

$\therefore$ By (*i*), OP = OQ *i.e.,* OQ = OP

$\therefore$ By (*i*), $(Q, P) \in R$. $\therefore$ R is symmetric.

Is R transitive? Let points $P, Q, S \in A$ such that $(P, Q) \in R$ and $(Q, S) \in R$.

$\therefore$ By (*i*), OP = OQ and OQ = OS. $\therefore$ OP = OS (= OQ)

$\therefore$ By (*i*), $(P, S) \in R$ $\therefore$ R is transitive.

Hence R is an equivalence relation.

Now given point P $\neq$ (0, 0).

Let Q be any point of set A related to point P.

$\therefore$ By (*i*), OQ = OP for all points $Q \in A$, related to point P.

i.e., OQ = constant distance OP = k (say)

$\therefore$ By definition of circle, all points Q of set A related to a given point P of A lie on a circle with centre at the origin O.

12. Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2) : T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Sol. Given: Relation R defined in the set (say A) of all triangles as

R = {(T_1, T_2) : T_1 is similar to T_2} ...(*i*)

We know that two triangles are said to be similar if the ratio of their corresponding sides is same. ...(*ii*)

Is R reflexive? Let $T_1 \in A$.

Putting $T_2 = T_1$ in (*i*), we have triangle T_1 is similar to triangle T_1 which is always true.

Therefore $(T_1, T_1) \in R$ for all $T_1 \in A$. $\therefore$ R is reflexive.

Is R symmetric? Let $T_1 \in A$, $T_2 \in A$ such that $(T_1, T_2) \in R$.

$\therefore$ By (*i*), triangle T_1 is similar to triangle T_2.

$\therefore$ We can say that triangle T_2 is similar to triangle T_1.

$\therefore$ By (*i*), $(T_2, T_1) \in R$. $\therefore$ R is symmetric.

Is R transitive? Let $T_1 \in A$, $T_2 \in A$, $T_3 \in A$ such that $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$.

By (*i*), $(T_1, T_2) \in R \Rightarrow T_1$ is similar to T_2

and $(T_2, T_3) \in R \Rightarrow T_2$ is similar to T_3.

$\therefore$ By definition of similar triangles given above (in (*ii*)),

Triangle T_1 is similar to triangle T_3.

$\therefore$ By (*i*), $(T_1, T_3) \in R$.

$\therefore$ R is transitive.

Hence R is an equivalence relation.

Now given that triangle T_1 has sides 3, 4, 5.............................. (*iii*)

Triangle T_2 has sides 5, 12, 13.. (*iv*)

Triangle T_3 has sides 6, 8, 10.. (*v*)

Triangle T_1 in (*iii*) is not similar to triangle T_2 in (*iv*)

($\because$ Ratio of their corresponding sides is not same *i.e.*, $\frac{3}{5}, \frac{4}{12}, \frac{5}{13}$ are not equal.)

$\therefore$ By (*i*), T_1 is not related to T_2.

Triangle T_2 in (*iv*) is not similar to triangle T_3 in (*v*) because ratio of their corresponding sides is not same *i.e.*, $\frac{5}{6}, \frac{12}{8}$ and $\frac{13}{10}$ are not equal.

$\therefore$ By (*i*), T_2 is not related to T_3.

But triangle T_1 in (*iii*) is similar to triangle T_3 in (*v*).

[$\because$ Ratio of their corresponding sides is same

i.e., $\frac{3}{6} = \frac{4}{8} = \frac{5}{10} \left(= \frac{1}{2}\right)$]

$\therefore$ By (i), triangle T_1 is related to triangle T_3.

13. Show that the relation R defined in the set A of all polygons as R = {(P_1, P_2) : P_1 and P_2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

Sol. Given: Relation R defined in the set A of all polygons as

$R = \{(P_1, P_2) : P_1$ and P_2 have same number of sides$\}$ ...(i)

Is R reflexive? Let polygon $P_1 \in A$.

Putting $P_2 = P_1$ in (i), we have P_1 and P_1 have same number of sides which is of course true.

($\because$ P_1 and P_1 is same polygon) $\therefore$ R is reflexive.

Is R symmetric? Let $P_1 \in A$ and $P_2 \in A$ such that $(P_1, P_2) \in R$.

$\therefore$ By (i), polygons P_1 and P_2 have the same number of sides, *i.e.,* polygons P_2 and P_1 have the same number of sides.

$\therefore$ By (i), $(P_2, P_1) \in R$. $\therefore$ R is symmetric.

Is R transitive? Let $P_1, P_2, P_3 \in A$ such that $(P_1, P_2) \in R$ and $(P_2, P_3) \in R$.

$\therefore$ By (i), polygons P_1 and P_2 have same number of sides.

Also polygons P_2 and P_3 have same number of sides.

$\therefore$ Polygons P_1 and P_3 (also) have same number of sides.

$\therefore$ By (i), $(P_1, P_3) \in R$. $\therefore$ R is transitive.

$\therefore$ R is an equivalence relation.

Now given a right angled triangle T with sides 3, 4, 5.

$\therefore$ T also $\in$ A ($\because$ A triangle is a polygon with three sides)

$\therefore$ By (i), the set of **all** triangles (polygons) of set A (and not only right angled triangles $\in$ A) is the set of all elements related to this given right angled triangle T.

14. Let L be the set of all lines in XY plane and R be the relation in L defined as R = {(L_1, L_2) : L_1 is parallel to L_2}. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Sol. Given: L = set of all the lines in XY plane.

Relation $R = \{(L_1, L_2) : L_1$ is parallel to $L_2\}$ ...(i)

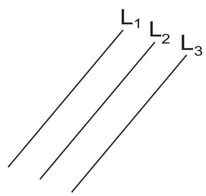
Is R reflexive?

Let $L_1 \in A$. Putting $L_2 = L_1$ in (i), we have L_1 is parallel to L_1 which is true.

$\therefore$ $(L_1\ L_1) \in R$. $\therefore$ R is reflexive.

Is R symmetric?

Let $L \in A$, $L \in A$ such that $(L, L) \in R$.



1 2 1 2

$\therefore$ By (i), line L_1 is parallel to line L_2.

$\therefore$ Line L_2 is parallel to line L_1.

$\therefore$ By (i), $(L_2, L_1) \in R$.

$\therefore$ R is symmetric.

Is R transitive?

Let $L_1 \in A$, $L_2 \in A$, $L_3 \in A$ such that $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$.

$\therefore$ By (i), line L_1 is parallel to L_2 and L_2 is parallel to L_3.

Hence L_1 is parallel to L_3.

$\therefore$ By (i), $(L_1, L_3) \in R$. $\therefore$ R is transitive.

Hence R is an equivalence relation.

Now to find set of all lines related to the line $y = 2x + 4$.

We know that equations of parallel lines differ only in constant term.

$\therefore$ By (i), (Equation), *i.e.,* set of all lines parallel to the line $y = 2x + 4$ is $y = 2x + c$ where c is an arbitrary constant.

15. Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation.

Sol. Given: Set is {1, 2, 3, 4} = A (say)

Relation R in A is {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)} ...(i)

R is reflexive because (1, 1), (2, 2), (3, 3), (4, 4) all belong to R *i.e.,* $(a, a) \in R$ for all $a \in A$.

Now $(1, 2) \in R$ but $(2, 1) \notin R$. $\therefore$ R is not symmetric.

Is R transitive?

Yes, because in relation (i) whenever (x, y) and $(y, z) \in R$; then (x, z) also $\in R$.

As for (1, 2) and $(2, 2) \in R$; $(1, 2) \in R$

For (1, 3) and $(3, 2) \in R$; $(1, 3) \in R$

$\therefore$ R is transitive. Therefore, option (B) is the correct option.

16. Let R be the relation in the set N given by R = {(a, b) : $a = b - 2$, $b > 6$}. Choose the correct answer.

(A) $(2, 4) \in R$ (B) $(3, 8) \in R$ (C) $(6, 8) \in R$ (D) $(8, 7) \in R$.

Sol. Relation R in **N** defined by $R = \{(a, b), a = b - 2, b > 6\}$ ...(i)

Out of the given options only option (C) satisfies (i) *i.e.,* $(6, 8) \in R$.

$\because$ $a = 6$, $b = 8 > 6$ and $8 - 2 = 6$ *i.e.,* $a = b - 2$.

Exercise 1.2

1. **Show that the function $f : \mathbf{R}_* \rightarrow \mathbf{R}_*$, defined by $f(x) = \frac{1}{x}$ is one-one and onto, where $\mathbf{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R}_*$ is replaced by N with co-domain being same as $\mathbf{R}_*$?**

Sol. $f(x) = \frac{1}{z}$ (given)

$\therefore$ For $x_1, x_2 \in R_*$ with $f(x_1) = f(x_2)$ we have $\frac{1}{z_1} = \frac{1}{z_2}$ $\left(\because f(z) = \frac{1}{z}\right)$

$\Rightarrow x_1 = x_2$ $\therefore$ f is one-one.

Again $f(x) = y = \frac{1}{z}$ $\Rightarrow$ $xy = 1$ $\Rightarrow$ $x = \frac{1}{y}$.

$\therefore$ Given any $y \in R_*$, the co-domain, we can choose $x = \frac{1}{y}$ such that

$f(x) = \frac{1}{z} = \frac{1}{\left(\frac{1}{y}\right)} = y$. Thus, every element in the co-domain has pre-image in the domain under f. Therefore, f is onto.

The function $f : N \rightarrow R_*$ again defined by $f(x) = \frac{1}{z}$ is also one-one,

since $x_1, x_2 \in N$ with $f(x_1) = f(x_2)$ $\Rightarrow$ $\frac{1}{z_1} = \frac{1}{z_2} \Rightarrow x_1 = x_2$.

But f is not onto since $2 \in R_*$ and 2 is not the reciprocal of any natural number, *i.e.,* there exists no natural number x such that $f(x) = 2$ or $\frac{1}{z} = 2$ or $x = \frac{1}{2}$ because $x = \frac{1}{2} \notin N$.

2. Check the injectivity and surjectivity of the following functions:

(i) $f : \mathrm{N} \to \mathrm{N}$ **given by** $f(x) = x^2$

(ii) $f : \mathrm{Z} \to \mathrm{Z}$ **given by** $f(x) = x^2$

(iii) $f : \mathrm{R} \to \mathrm{R}$ **given by** $f(x) = x^2$

(iv) $f : \mathrm{N} \to \mathrm{N}$ **given by** $f(x) = x^3$

(v) $f : \mathrm{Z} \to \mathrm{Z}$ **given by** $f(x) = x^3$

Sol. **Note.** Checking **injectivity** $\Rightarrow$ Testing for **one-one** function checking **surjectivity** $\Rightarrow$ Testing for **onto** function.

(i) **Given:** $f : \underset{\text{domain}}{\underset{\downarrow}{\mathrm{N}}} \to \underset{\text{co-domain}}{\underset{\downarrow}{\mathrm{N}}}$ given by $f(x) = x^2$ ...(i)

Is f injective (one-one)? Let $x_1, x_2 \in$ Domain N such that $f(x_1) = f(x_2)$

Putting $x = x_1$ and $x = x_2$ in (i), $x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

Rejecting negative sign ($\because$ $x_1, x_2 \in$ N and hence are positive), $x_1 = x_2$

$\therefore$ f is injective *i.e.,* one-one.

Is f surjective (onto)?

Putting $x = 1, 2, 3, ..., \in$ domain N, in (i),
Range set = $\{f(x) = x^2; x \in N\} = \{1^2, 2^2, 3^2, ...\} = \{1, 4, 9, ...\}$
$\neq$ co-domain N
[$\because$ 2, 3, 5, 8, ... $\in$ co-domain N but don't belong range]
$\therefore$ f is not surjective.
$\therefore$ f is injective but not surjective.

(ii) **Given:** $f : Z \to Z$ given by $f(x) = x^2$ ...(i)

$\downarrow$ domain, $\downarrow$ co-domain

Is f injective (one-one)? Let $x_1, x_2 \in$ Domain Z such that $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$ (By (i))
$\Rightarrow x_1 = \pm x_2$
Now negative sign can't be rejected because $x_1, x_2 \in$ Z (set of integers) can be negative also.
$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = \pm x_2$.
Hence f is not one-one.
(Also for example $2 \in Z, -2 \in Z, 2 \neq -2$ but by (i),
$f(2) = 2^2 = 4 = f(-2) = (= (-2)^2 = 4))$.

Is f onto?
Putting $x = 0, \pm 1, \pm 2, \pm 3 ..., \in$ domain Z, in (i),
Range set = $\{f(x) = x^2; x \in Z\} = \{0, 1, 4, 9, ...\}$
$\neq$ co-domain Z
[$\because$ 2, – 2, etc. belong to co-domain Z but don't belong to range]
$\therefore$ f is not surjective. (*i.e.,* not onto)
$\therefore$ f is neither injective nor surjective.

(iii) **Given:** $f : R \to R$ given by $f(x) = x^2$ ...(i)

$\downarrow$ domain, $\downarrow$ co-domain

Is f injective? [Replace Z by R in the solution for part (ii) namely "Is f injective".]
Is f surjective?
Range set = $\{f(x) = x^2; x \in R\}$ is the set of all positive real numbers only including 0.
($\because$ $x^2 \geq 0$ for all $x \in R$) and hence $\neq$ co-domain R.
$\therefore$ f is not surjective.
$\therefore$ f is neither injective nor surjective.

(iv) **Given:** $f : N \to N$ given by $f(x) = x^3$ ...(i)

$\downarrow$ domain, $\downarrow$ co-domain

Is f injective? Let $x_1, x_2 \in$ domain N such that $f(x_1) = f(x_2)$
$\therefore$ By (i), $x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ (only). $\therefore$ f is one-one (injective).
Is f surjective ?

Putting $x = 1, 2, 3, \ldots \in$ domain N, in (*i*),

Range set $= \{f(x) = x^3;\ x \in \text{N}\} = \{1^3, 2^3, 3^3, 4^3, \ldots\}$

$= \{1, 8, 27, 64, \ldots\} \neq$ co-domain N

($\because$ 2, 3, 5, 8 etc. belong to co-domain N but don't belong to range set)

$\therefore$ f is not surjective.

Hence f is injective but not surjective.

(v) **Given:** $f: \underset{\text{domain}}{\text{Z}} \to \underset{\text{co-domain}}{\text{Z}}$ given by $f(x) = x^3$ ...(*i*)

Is f injective? Let $x_1, x_2 \in$ domain Z such that $f(x_1) = f(x_2)$

$\therefore$ By (*i*), $x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ (only).

$\therefore$ f is one-one (injective).

Is f surjective?

Putting $x = 0, 1, -1, 2, -2, \ldots \in$ domain Z, in (*i*),

Range set $= \{f(x) = x^3;\ x \in \text{Z}\} = \{0, 1, -1, 8, -8, \ldots\} \neq$ co-domain Z.

($\because$ 2, − 2, 3, − 3, ... belong to co-domain Z but don't belong to range set)

$\therefore$ f is not surjective. $\therefore$ f is injective but not surjective.

3. Prove that the Greatest Integer Function f : R → R, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Sol. Given: Greatest integer function

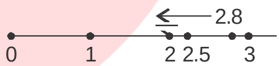
f : R → R given by $f(x) = [x]$ ...(*i*)

where $[x]$ denotes the **greatest integer less than** or **equal to x.**

Is f one-one? Let us take $x_1 = 2.5$ and $x_2 = 2.8 \in$ domain R.

From (*i*), $f(x_1) = f(2.5) = [2.5] = 2$

and $f(x_2) = f(2.8) = [2.8] = 2$



$\therefore$ $f(x_1) = f(x_2)$ (= 2) but x_1 (= 2.5) $\neq x_2$ (= 2.8)

$\therefore$ f is not one-one.

Is f onto? By (*i*), range set $= \{f(x) = [x],\ x \in \text{R}\}$ is the set Z of all integers ($\because$ value of $[x]$ is always an integer)

But $\neq$ co-domain R.

$\therefore$ f is not onto.

$\therefore$ f is neither one-one nor onto.

4. Show that the Modulus Function f : R → R, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Sol. Given: Modulus function f : R → R given $f(x) = |x|$ ...(*i*)

Is f one-one? Let us take $x_1 = -1$ and $x_2 = 1 \in$ domain R

From (*i*), $f(x_1) = f(-1) = |-1| = 1$

and $f(x_2) = f(1) = |1| = 1$

$\therefore$ $f(x_1) = f(x_2)$ $(= 1)$ but $x_1(= -1) \neq x_2$ $(= 1)$

$\therefore$ f is not one-one.

Is f onto?

By (i), Range set $= \{ f(x) = |x|, x \in R\}$ is the set of positive real numbers (only) including 0. $(\because \ |x| \geq 0$ for all $x \in R)$ but $\neq$ co-domain R.

$\therefore$ f is not onto. $\therefore$ f is neither one-one nor onto.

5. Show that the Signum Function $f: R \to R$, given by

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

Sol. Is f one-one?

Let $x_1 = 2, x_2 = 3$ $\therefore$ $f(x_1) = f(2) = 1$ and $f(x_2) = f(3) = 1$

$[\because \ 2 > 0, 3 > 0$ and it is given that $f(x) = 1$ for $x > 0]$

$\therefore$ $f(x_1) = f(x_2) = 1$ but x_1 $(= 2) \neq x_2$ $(= 3)$ $\therefore$ f is not one-one.

or Let $x_1 = -2, x_2 = -3$

$\therefore$ $f(x_1) = f(-2) = -1$ and $f(x_2) = f(-3) = -1$

$[\because \ -2 < 0, -3 < 0$ and it is given that $f(x) = -1$ for $x < 0]$

$\therefore$ $f(x_1) = f(x_2) = -1$ but x_1 $(= -2) \neq x_2$ $(= -3)$

$\therefore$ f is not one-one.

Is f onto?

According to given $f(x) = 1$ if $x > 0$, $= 0$ if $x = 0$ and $= -1$ if $x < 0$.

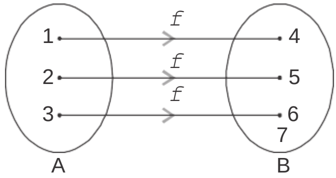
$\therefore$ Range set $= \{f(x) : x \in R\} = \{1, 0, -1\} \neq$ co-domain R.

$\therefore$ f is not onto. $\therefore$ f is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

Sol. Given: set A = {1, 2, 3}, B = {4, 5, 6, 7} and

f = {(1, 4), (2, 5), (3, 6)} is a function from A to B.



$\Rightarrow$ $f(1) = 4$, $f(2) = 5$, $f(3) = 6$.

$\Rightarrow$ Distinct elements 1, 2 and 3 of domain A have distinct f-images 4, 5, 6 in co-domain B.

Therefore f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3 - 4x$.

(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 + x^2$.

Sol. (*i*) **Given:** f : R $\to$ R defined by $f(x) = 3 - 4x$ ...(*i*)

$\downarrow$ domain $\quad \downarrow$ co-domain

Is f one-one? Let $x_1, x_2 \in$ domain R such that

$f(x_1) = f(x_2)$ $\Rightarrow$ $3 - 4x_1 = 3 - 4x_2$

[Putting $x = x_1$ and $x = x_2$ in (*i*), $f(x_1) = 3 - 4x_1$ and $f(x_2) = 3 - 4x_2$] $\Rightarrow$ $-4x_1 = -4x_2$

Dividing by -4, $x_1 = x_2$ $\therefore$ f is one-one.

Is f onto? Let y be any element of co-domain R.

Let us suppose that $y = f(x)$

$\therefore$ $y = 3 - 4x$ (By (*i*)) ...(*ii*)

Let us find x in terms of y.

From (*ii*), $4x = 3 - y$ $\Rightarrow$ $x = \dfrac{3-y}{4}$

Now $x = \dfrac{3-y}{4} \in$ domain R for each $y \in$ co-domain.

$\therefore$ By (*i*), $f(x) = 3 - 4x = 3 - 4\left(\dfrac{3-y}{4}\right) = 3 - (3 - y)$

$|\left(4 \right)|$

$= 3 - 3 + y = y$

Hence our supposition that $y = f(x)$ is correct.

Hence co-domain = Range

$\therefore$ f is onto. $\therefore f$ is both one-one and onto, *i.e.*, f is bijective.

(*ii*) **Given:** $f : R \to R$ defined by $f(x) = 1 + x^2$ ...(*i*)

Is f one-one? Let $x_1, x_2 \in$ domain R such that $f(x_1) = f(x_2)$.

$\therefore$ From (*i*), $1 + x_1^2 = 1 + x_2^2 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

Negative sign can't be rejected because $x_1, x_2 \in$ R can be negative.

$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = \pm x_2$ and not $x_1 = x_2$.

Hence f is not one-one.

[For example, from (*i*), $f(1) = 1 + 1^2 = 1 + 1 = 2$

and $f(-1) = 1 + (-1)^2 = 1 + 1 = 2$.

Now $1 \neq -1$ but $f(1) = f(-1)$]

Is f onto?

Range set $= \{ f(x) = 1 + x^2 : x \in R\}$

$= \{ f(x) \geq 1$ ($\because$ $x^2 \geq 0$ always)$\}$

$= [1, \infty) \neq$ co-domain R $(= (-\infty, \infty))$

$\therefore$ f is not onto. $\therefore$ f is neither one-one nor onto.

8. Let A and B be sets. Show that f : A × B → B × A such that $f(a, b) = (b, a)$ is bijective function.

Sol. Let $(a, b), (c, d) \in A \times B$ with $f(a, b) = f(c, d)$, then $(b, a) = (d, c)$

[$\because$ The given rule is $f(a, b) = (b, a)$]

Equating corresponding entries $b = d$ and $a = c$

$\Rightarrow (a, b) = (c, d)$ $\therefore$ f is injective (one-one) function. ....(*i*)

Also, for all $(b, a) \in$ co-domain B × A, there exists $(a, b) \in$ domain A × B such that by the given rule $f(a, b) = (b, a)$. Thus every $(b, a) \in B \times A$ is the image of $(a, b) \in A \times B$.

$\therefore f$ is surjective (onto) function. .... (*ii*)

From (*i*) and (*ii*), f is a one-one onto *i.e.*, bijective function.

9. Let f : N → N be defined by

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in N.$$

State whether the function f is onto, one-one or bijective. Justify your answer.

Sol. $f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if} \end{cases}$

2

n is
even

To examine whether f is one-one.

Now $3 \in$ N and $4 \in$ N and $3 \neq 4$

$3 \in$ N is odd, $\therefore$ from (i), $f(3) = \frac{3+1}{2} = 2$

$4 \in$ N is even, $\therefore$ from (ii), $f(4) = \frac{4}{2} = 2$

$\therefore f(3) = f(4)$ but $3 \neq 4$ $\therefore f$ is not one-one.

To examine whether f is onto.

Let us find range of f.

Putting $n = 1, 3, 5, \ldots\ 2n - 1, \ldots$ (all odd) in (i),

$$f(1) = \frac{1+1}{2} = 1,\ f(3) = \frac{3+1}{2} = 2,\ f(5) = \frac{5+1}{2} = 3, \ldots$$

$$f(2n-1) = \frac{2n-1+1}{2} = n, \ldots$$

Putting $n = 2, 4, 6, \dots 2n, \dots$ (all even) in (*ii*),

$$f(2) = \frac{2}{2} = 1, f(4) = \frac{4}{2} = 2, f(6) = \frac{6}{2} = 3,$$

$$f(2n) = \frac{2n}{2} = n, \dots\dots$$

$\therefore$ Range of $f = \{1, 2, 3, 4, \dots, n, \dots\}$ = co-domain N

$\therefore$ f is onto. $\therefore$ f is not one-one but onto

Hence, f is not bijective.

Note. While discussing **onto,** whenever for $y \in$ co-domain, there are two or more than two $x \in$ domain, for which $f(x) = y$. Then f is never one-one.

Here in the above example for $y = 1 \in$ co-domain N, there are two values of x (*i.e.,* two pre-images) $x = 1$, $x = 2 \in$ domain N for which $f(x) = y$

$\therefore$ f is not one-one.

10. Let A = R – {3} and B = R – {1}. Consider the function f : A → B defined by $f(x) = \frac{x-2}{x-3}$. Is f one-one and onto? Justify your answer.

Sol. For $x_1, x_2 \in$ A with $f(x_1) = f(x_2) \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$$\left(\text{On putting } x = x_1 \text{ and } x = x_2 \text{ in } f(x) = \frac{x-2}{x-3}\right)$$

Cross-multiplying, $(x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow -3x_1 - 2x_2 = -2x_1 - 3x_2 \Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\therefore$ f is one-one.

Now let $y = f(x) = \frac{x-2}{x-3}$, $\Rightarrow (x - 3)y = x - 2$

$\Rightarrow xy - 3y = x - 2$ $\Rightarrow xy - x = 3y - 2$

$\Rightarrow x(y - 1) = 3y - 2$ $\Rightarrow x = \frac{3y-2}{y-1}$

$\therefore$ For every $y \neq 1$, *i.e.,* when $y \in$ co-domain B, there exists $x = \frac{3y-2}{y-1} \in A = R - \{3\}$ such that

$$f(x) = \frac{x-2}{x-3} = \frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3} = \frac{y-1}{}$$

$\frac{3y-2-2y+2}{3y-2-3y+3} = y$

$\therefore$ f is onto.

11. Let $f: \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto **(B) f is many-one onto**

(C) f is one-one but not onto

(D) f is neither one-one nor onto.

Sol. $f : R \to R$ is defined as $f(x) = x^4$ ...(i)

Is f one-one? No

Because 2 and – 2 both belong to domain R, but by (i),

$f(2) = 2^4 = 16$ and $f(-2) = (-2)^4 = 16$

$\therefore$ $f(2) = f(-2)$ but $2 \neq -2$

($i.e.,$ $f(x_1) = f(x_2)$ but $x_1 \neq x_2$)

Is f onto? No

Because range set $= \{f(x) = x^4, x \in R\}$

$= \{f(x) \geq 0\}$ ($\because$ $x^4 \geq 0$ for all real x)

$= [0, \infty) \neq$ co-domain R $(= (-\infty, \infty))$

$\therefore$ f is not onto. $\therefore$ f is neither one-one nor onto.

$\therefore$ Option (D) is the correct answer.

12. Let $f : R \to R$ be defined as $f(x) = 3x$. Choose the correct answer.

(A) f is one-one onto **(B) f is many-one onto**

(C) f is one-one but not onto

(D) f is neither one-one nor onto.

Sol. $f : R \to R$ is defined as $f(x) = 3x$ ...(i)

Is f one-one?

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$. $\therefore$ By (i), $3x_1 = 3x_2$

Dividing by 3, $x_1 = x_2$ $\therefore$ f is one-one.

Is f onto?

Let $y \in$ co-domain R.

Let $y = f(x)$ $\therefore$ $y = 3x$ (By (i))

$\therefore$ $x = \dfrac{y}{3} \in$ domain R for every $y \in$ co-domain R

$\therefore$ By (i), $f(x) = 3x = 3\left(\dfrac{y}{3}\right) = y$ $i.e.,$ Range = co-domain.

$\therefore$ f is onto. $\therefore$ f is both one-one and onto.

$\therefore$ Option (A) is the correct answer.

Exercise 1.3

1. Let $f : \{1, 3, 4\} \to \{1, 2, 5\}$ and $g : \{1, 2, 5\} \to \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down *gof*.

Sol. Let A = $\{1, 3, 4\}$, B = $\{1, 2, 5\}$ and C = $\{1, 3\}$.

Given: $f : A \to B$ and $g : B \to C \Rightarrow gof : A \to C$

i.e., $gof : \{1, 3, 4\} \to \{1, 3\}$.

Now $f = \{(1, 2), (3, 5), (4, 1)\} \Rightarrow f(1) = 2, f(3) = 5, f(4) = 1$

Also $g = \{(1, 3), (2, 3), (5, 1)\} \Rightarrow g(1) = 3, g(2) = 3, g(5) = 1$

$\therefore \quad (gof)(1) = g(f(1)) = g(2) = 3,$

$(gof)(3) = g(f(3)) = g(5) = 1$ and $(gof)(4) = g(f(4)) = g(1) = 3$

Hence, $gof = \{(1, 3), (3, 1), (4, 3)\}$.

2. Let *f*, *g* and *h* be functions from R to R. Show that

(*i*) $(f + g)oh = foh + goh$ (*ii*) $(fg)oh = (foh) . (goh)$

Sol. Definition of Equal functions

Two functions f and g are said to be equal if they have the same domain D and $f(x) = g(x)$ for all $x \in$ D.

(i) $[(f + g)oh](x) = (f + g)(h(x))$

(By Def. of composite function)

$= f(h(x)) + g(h(x))$

(By Def. of sum function)

$= (foh)(x) + (goh)(x)$

(By Def. of composite function)

$= (foh + goh)x$ (By Def. of sum function) for all $x \in$ R, domain.

$\Rightarrow (f + g)oh = foh + goh$ (By Def. of equal functions)

(ii) $[(fg)oh](x) = (fg)(h(x))$ (By Def. of composite function)

$= f(h(x)) . g(h(x))$ (By Def. of product function)

$= (foh)(x) . (goh)(x)$ (By Def. of composite function)

$= ((foh) . (goh))x$ (By Def. of product function)

$\Rightarrow (fg)oh = (foh) . (goh)$. (By Def. of equal functions)

3. Find *gof* and *fog* if

(i) $f(x) = |x|$ and $g(x) = |5x - 2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{1/3}$.

Sol. (*i*) **Given:** $f(x) = |x|$ ...(*i*)

and $g(x) = |5x - 2|$ ...(*ii*)

We know that $\boldsymbol{(gof)x = g(f(x))} = g(|x|)$ [By (*i*)]

Changing $x \to |x|$ in (*ii*), $= |5|x| - 2|$

Again $(fog)x = f(g(x)) = f(|5x - 2|)$ [By (*ii*)]

Changing x to $|5x - 2|$ in (*i*) $= ||5x - 2|| = |5x - 2|$

[$\because$ $t \in$ R $\Rightarrow |t| \geq 0$ and hence $||t||$ is number $|t|$ itself]

$\therefore \quad (gof)x = |5|x| - 2|$ and $(fog)x = |5x - 2|$.

(*ii*) **Given:** $f(x) = 8x^3$ ...(*i*) and $g(x) = x^{1/3}$ ...(*ii*)

We know that $(gof)x = g(f(x)) = g(8x^3)$ (By (*i*))

Changing x to $8x^3$ in (*ii*), $= (8x^3)^{1/3} = ((2x)^3)^{1/3} = 2x$

Again $(fog)x = f(g(x)) = f(x^{1/3})$ (By (*ii*))

Changing x to $x^{1/3}$ in (*i*), $= 8\ (x^{1/3})^3 = 8x$

$\therefore$ $(gof)x = 2x$ and $(fog)x = 8x$.

4. If $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$, for all $x \neq \dfrac{2}{3}$. What is the inverse of f?

Sol. Here $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq$ 2 ...(*i*)

$$\therefore \quad fof(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right) \qquad \text{[by } (i)\text{]}$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} \qquad \left[\text{Replacing } x \text{ by } \frac{4x+3}{6x-4} \text{ in } (i)\right]$$

Multiplying by L.C.M. = $(6x - 4)$

$$(fof)\,x = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x = I_A(x) \text{ for all } x \neq \frac{2}{3}$$

$\Rightarrow \quad fof = I_A$ where $A = R - \left\{\frac{2}{3}\right\}$ = domain of f

$\Rightarrow \quad f^{-1} = f.$

5. State with reason whether following functions have inverse. Find the inverse, if it exists.

(i) $f : \{1, 2, 3, 4\} \to \{10\}$ with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(ii) $g : \{5, 6, 7, 8\} \to \{1, 2, 3, 4\}$ with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(iii) $h : \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$ with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

Sol. (*i*) Since $f(1) = f(2) = f(3) = f(4) = 10$, f is many-one and not one-one, so that f does not have inverse.

(ii) Since $g(5) = g(7) = 4$, g is many-one and not one-one, so that g does not have inverse.

(iii) Here distinct elements of the domain have distinct images under h, so that h is one-one. Moreover range of $h = \{7, 9, 11, 13\}$ = co-domain, so that h is onto.

Since h is one-one and onto, therefore, h has inverse given by $h^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$.

6. Show that $f : [-1, 1] \to R$, given by $f(x) = \dfrac{x}{x+2}$ is one-one.

Find the inverse of the function $f : [-1, 1] \to$ Range f.

Sol. For $x_1, x_2 \in [-1, 1]$.

Putting $x = x_1$ and $x = x_2$ in $f(x) = \dfrac{x}{x+2}$,

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1+2} = \frac{x_2}{x_2+2}$$

Cross-multiplying,

$$x_1(x_2 + 2) = x_2(x_1 + 2)$$

$\Rightarrow \quad x_1x_2 + 2x_1 = x_1x_2 + 2x_2$

$\Rightarrow \quad 2x_1 = 2x_2 \quad \Rightarrow x_1 = x_2 \qquad \therefore f$ is one-one.

Now we have to find the inverse of the function $f : X \to Y$ (say)

= Range of f (given)

where $X = [-1, 1]$

and $\quad Y = \left\{ y \in R : y = \dfrac{x}{x+2} \text{ for some } x \in X \right\}$ = range of f.

Therefore, f is onto.

[$\because$ Co-domain Y = Range of f (given)]

Thus, f is one-one and onto and therefore, f^{-1} exists.

$\therefore \quad y = f(x) \quad \Rightarrow \quad x = f^{-1}(y)$ ...(*i*)

To find f^{-1}, let us find x in terms of y.

Given: $y = \dfrac{x}{x+2}$

Cross-multiplying,

$xy + 2y = x$ or $2y = x(1 - y)$ or $x = \dfrac{2y}{1-y}, y \neq 1$ ...(*ii*)

From (*i*) and (*ii*),

$\therefore f^{-1} : Y \to X$ is defined by $f^{-1}(y) (= x) = \dfrac{2y}{1-y}, y \neq 1.$

7. Consider $f : R \to R$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

Sol. Given: $f : R \to R$ given by $f(x) = 4x + 3$ ...(*i*)

We know that a function f is invertible iff f is one-one and onto.

Is f one-one?

Let $x_1, x_2 \in$ domain R such that $f(x_1) = f(x_2)$

$\therefore$ By (*i*), $4x_1 + 3 = 4x_2 + 3 \Rightarrow 4x_1 = 4x_2$

Dividing by 4, $x_1 = x_2$. $\qquad \therefore f$ is one-one.

Is f onto? Let $y \in$ co-domain R.

Let $y = f(x) = 4x + 3$. (By (*i*))

Let us find x in terms of y.

$y = 4x + 3 \Rightarrow y - 3 = 4x$

$\Rightarrow \quad x = \dfrac{y-3}{4}$ ...(*ii*)

$\therefore$ By (*i*), $f(x) = 4x + 3 = 4\left(\dfrac{y-3}{4}\right) + 3$

$= y - 3 + 3 = y$

i.e., Every y is $f(x)$ *i.e.*, co-domain = Range $\quad \therefore \quad f$ is onto also.

$\therefore \quad f$ is both one-one and onto and hence f^{-1} exists.

$\therefore \quad f(x) = y \Rightarrow f^{-1}(y) = x \quad$ *i.e.*, $\quad f^{-1}(y) = \dfrac{y-3}{4}$. (By (*ii*))

8. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with inverse f^{-1} given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Sol. A function $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$ ....(i)

We know that a function f is invertible iff f is one-one and onto.

Is f one-one? Let $x_1, x_2 \in$ domain R_+ such that $f(x_1) = f(x_2)$.

$\therefore$ By (i), $x_1^2 + 4 = x_2^2 + 4$ or $x_1^2 = x_2^2$

$\therefore$ $x_1 = \pm x_2$

Rejecting negative sign ($\because$ $x_1, x_2 \in$ domain R_+ and hence can't be negative).

$\therefore$ $x_1 = x_2$.

$\therefore$ f is one-one.

Is f onto? Let $y \in$ co-domain $[4, \infty)$.

Let $y = f(x) = x^2 + 4$

Let us find x in terms of y. $y = x^2 + 4 \Rightarrow y - 4 = x^2$

$\Rightarrow$ $x^2 = y - 4 \Rightarrow x = \pm\sqrt{y-4}$

Rejecting negative sign ($\because$ $x \in R_+$),

$x = \sqrt{y-4}$ ...(ii)

and $x = \sqrt{y-4} \in R_+$ because for all $y \in$ co-domain $[4, \infty)$,

$y \geq 4$ *i.e.,* $y - 4 \geq 0$ and hence $\sqrt{y-4} \in R_+$

Putting $x = \sqrt{y-4}$ in $f(x) = x^2 + 4$, we have

$f(x) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$. Every $y = f(x)$

i.e., co-domain = Range *i.e.,* f is onto (also). $\therefore$ f^{-1} exists.

and $f(x) = y \Rightarrow f^{-1}(y) = x$ $\therefore$ $f^{-1}(y) = \sqrt{y-4}$ (By (ii))

9. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{(\sqrt{y+6}) - 1}{3}\right)$.

Sol. Given: $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$ ...(i)

We know that f is invertible iff f is one-one and onto.

Is f one-one? Let $x_1, x_2 \in R_+$ such that $f(x_1) = f(x_2)$.

$\therefore$ By (i), $9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow$ $9x_1^2 - 9x_2^2 + 6x_1 - 6x_2 = 0$

$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$\Rightarrow \quad (x_1 - x_2)(9x_1 + 9x_2 + 6) = 0$

But $9x_1 + 9x_2 + 6 > 0$ and hence $\neq 0$

$(\because \ x_1, x_2 \in R_+ \ \Rightarrow \ x_1 \geq 0$ and $x_2 \geq 0)$

$\therefore \ x_1 - x_2 = 0$ *i.e.*, $x_1 = x_2$ $\qquad \therefore$ f is one-one.

Is f onto? Let $y \in$ co-domain $[-5, \infty)$

Let $y = f(x) = 9x^2 + 6x - 5$ (By (i))

Let us find x in terms of y by completing squares on R.H.S.

$$y = 9x^2 + 6x - 5 = 9\left[x^2 + \frac{6x}{9} - \frac{5}{9}\right] = 9\left[x^2 + \frac{2x}{3} - \frac{5}{9}\right]$$

Adding and subtracting $\left(\frac{1}{2}\text{ coeff. of } x\right)^2 = \left(\frac{1}{2} \times \frac{2}{3}\right)^2 = \left(\frac{1}{3}\right)^2$

$$y = 9\left[x^2 + \frac{2x}{3} + \left(\frac{1}{3}\right)^2 - \frac{1}{9} - \frac{5}{9}\right] = 9\left[\left(x + \frac{1}{3}\right)^2 - \frac{6}{9}\right]$$

or $y = 9\left(x + \frac{1}{3}\right)^2 - 6 \ \Rightarrow \ y + 6 = 9\left(x + \frac{1}{3}\right)^2$

$$\Rightarrow \left(x + \frac{1}{3}\right)^2 = \frac{y+6}{9} \quad \Rightarrow \quad x + \frac{1}{3} = \pm\sqrt{\frac{y+6}{9}}$$

$$\Rightarrow \quad x = \frac{-1}{3} \pm \frac{\sqrt{y+6}}{3} = \frac{-1 \pm \sqrt{y+6}}{3}$$

Rejecting negative sign

$$\left(\because \text{ then } x = \frac{-1 - \sqrt{y+6}}{3} = -\left(\frac{1 + \sqrt{y+6}}{3}\right) \notin R_+\right)$$

$\therefore \ x = \dfrac{-1 + \sqrt{y+6}}{3} \in R_+$ for every $y \in$ co-domain $[-5, \infty)$ ....(ii)

$\therefore$ By (i), $f(x) = 9x^2 + 6x - 5$

$$= 9\left(\frac{-1 + \sqrt{y+6}}{3}\right)^2 + 6\left(\frac{-1 + \sqrt{y+6}}{3}\right) - 5$$

$\sqrt{y+6} \qquad \sqrt{}$

$$= (-1 + \sqrt{y+6})^2 + 2(-1 + \sqrt{y+6}) - 5$$

$$= 1 + y + 6 - 2\sqrt{y+6} - 2 + 2\sqrt{y+6} - 5$$

or $f(x) = y$ *i.e.,* every $y = f(x)$

i.e., co-domain = range. $\therefore$ f is onto (also).

$\therefore$ f^{-1} exists. $\therefore$ $y = f(x) \Rightarrow f^{-1}(y) = x$

i.e., $f^{-1}(y) = \dfrac{-1 + \sqrt{y+6}}{3}$ *i.e.,* $\dfrac{\sqrt{y+6} - 1}{3}$.

10. Let f : X $\to$ Y is an invertible function, then f has unique inverse.

Sol. ....f : X $\to$ Y is an invertible function, therefore f has an inverse.

Again because f : X $\to$ Y is invertible, therefore, f is one-one and onto.

i.e., for every $y \in$ Y, $\exists$ a unique $x \in$ X such that $f(x) = y$................(i)

If possible, let g_1 : Y $\to$ X and g_2 : Y $\to$ X be two inverses of f so that

$g_1(y) = x$ and $g_2(y) = x$.

Also $g_1of = I_X, fog_1 = I_Y$ ...(*ii*)

and also $g_2of = I_X, fog_2 = I_Y$ ...(*iii*)

For all $y \in Y$, $g_1(y) = g_1(f(x))$ [By (*i*)]

$= (g_1of)(x) = I_X(x)$ [By (*ii*)] $= (g_2of)(x)$ [By (*iii*)]

$= g_2(f(x)) = g_2(y)$ [By (*i*)]

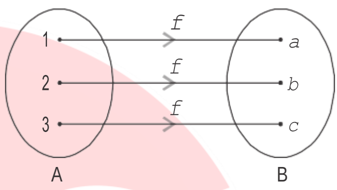
$\therefore$ $g_1(y) = g_2(y)$ for all $y \in Y$

$\Rightarrow$ $g_1 = g_2$ (By definition of equal functions).

11. Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Sol. Given: A function $f : \{1, 2, 3\}$ (= set A say) $\to \{a, b, c\}$ (= set B (say)) such that $f(1) = a$,

$f(2) = b, f(3) = c$ ...(*i*)



f is one-one because distinct elements of A have distinct images in B.

f is onto.

($\because$ Range set = $\{a, b, c\}$ = co-domain B)

$\therefore$ f^{-1} B $\to$ A exists and hence from (*i*),

$f^{-1}(a) = 1, f^{-1}(b) = 2$ and $f^{-1}(c) = 3$. .... (*ii*)

Now again by definition of Inverse of a function.

We have from (*ii*), $f^{-1} : A \to B$ and $(f^{-1})^{-1}(1) = a$,

$(f^{-1})^{-1}(2) = b$ and $(f^{-1})^{-1}(3) = c$.. (*iii*)

By definition of equal functions, from (*iii*) and (*i*), we have $(f^{-1})^{-1} = f$.

Note. Two functions f and g are said to be **equal** if they have (*i*) same domain and (*ii*), $f(x) = g(x)$ for all x in their common domain.

12. Let $f : X \to Y$ is an invertible function, then the inverse of f^{-1} is f, *i.e.*, $(f^{-1})^{-1} = f$.

Sol. Since $f : X \to Y$ is invertible, therefore, there exists a function

$g = f^{-1} : Y \to X$ defined by $g(y) = x$ where $f(x) = y$.

Also $gof = I_X$ and $fog = I_Y$ .... (*i*)

Now f is invertible

$\Rightarrow f$ is one-one and onto

$\Rightarrow g$, the inverse of f is also one-one and onto.

$\Rightarrow g$ is invertible

$\Rightarrow g^{-1}$ exists and $g^{-1} : X \to Y$.

From (*i*), we also have, f is inverse of g

$\Rightarrow f = g^{-1}$

Putting $g = f^{-1}$, we have $f = (f^{-1})^{-1}$

Hence $(f^{-1})^{-1} = f$.

13. If $f : R \to R$ be given by $f(x) = (3 - x^3)^{1/3}$ then $(fof)x$ is

(A) $x^{1/3}$ (B) x^3 (C) x (D) $(3 - x^3)$.

Sol. $f : R \to R$ is given by $f(x) = (3 - x^3)^{1/3}$ ...(i)

We know by definition that

$$(fof)x = f(f(x)) = f((3 - x^3)^{1/3}) \quad \text{(By (i))}$$
$$= f(t) \text{ where } t = (3 - x^3)^{1/3}$$
$$= (3 - t^3)^{1/3} \quad \text{(By (i))}$$

Putting $t = (3 - x^3)^{1/3}$; $= (3 - ((3 - x^3)^{1/3})^3)^{1/3}$

$$= (3 - (3 - x^3))^{1/3}. \quad \left(\because 3 \times \frac{1}{3} = 1\right)$$

$$= (3 - 3 + x^3)^{1/3} = (x^3)^{1/3} = x^{3 \times 1/3} = x^1 = x$$

$\therefore$ Option (C) is the correct answer.

14. Let $f : R - \left\{-\frac{4}{3}\right\} \to R$ be a function defined as $f(x) = \frac{4x}{3x+4}$.

The inverse of f is the map g : Range $f \to R - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$ **(B) $g(y) = \frac{4y}{4-3y}$**

(C) $g(y) = \frac{4y}{3-4y}$ **(D) $g(y) = \frac{3y}{4-3y}$.**

Sol. Given: $f(x) = \frac{4x}{3x+4}$ ...(i)

Given: Inverse of f is g. Therefore f is one-one and onto.

$$\therefore \quad y = f(x) = \frac{4x}{3x+4}. \quad \text{(By (i))}$$

Let us find x in terms of y.

Cross-multiplying, $y(3x + 4) = 4x \Rightarrow 3xy + 4y = 4x$

$\Rightarrow$ $-4x + 3xy = -4y \Rightarrow -x(4 - 3y) = -4y$

$\Rightarrow$ $x = \frac{4y}{4-3y} \Rightarrow f^{-1}(y) = \frac{4y}{4-3y}$

$[\because y = f(x) \Rightarrow x = f^{-1}(y)]$

$\Rightarrow g(y) = \frac{4y}{4-3y}$ $[\because g = f^{-1}$ (given)$]$

$\therefore$ Option (B) is correct option.

Exercise 1.4

1. **Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.**
   - **(i) On Z^+, define $*$ by $a * b = a - b$**
   - **(ii) On Z^+, define $*$ by $a * b = ab$**
   - **(iii) On R, define $*$ by $a * b = ab^2$**
   - **(iv) On Z^+, define $*$ by $a * b = |a - b|$**
   - **(v) On Z^+, define $*$ by $a * b = a$.**

Sol. Definition: Binary operation $*$ on a set A is a function from $A \times A \to A$.

i.e., an operation $*$ on a set A is called binary operation if $a \in A$ and $b \in A \Rightarrow a * b \in A$.

We also know that Z^+ denotes the set of positive integers

i.e., **$Z^+ = N$.**

(i) **Given:** On Z^+ define $*$ by $a * b = a - b$.. (*i*)

Now $a = 2 \in Z^+$ and $b = 3 \in Z^+$, but $a * b = 2 * 3$

$= 2 - 3$ (By (*i*)) $= -1 \notin Z^+$

$\because$ Operation $*$ in (*i*) is **not** a binary operation and hence does not satisfy **closure law.**

Remark. It may be noted for the above (*i*) part that if $a \in Z^+$, $b \in Z^+$ and $a > b$, then $a * b = a - b$ is a positive integer and hence belongs to Z^+.

But let us know that truth is always truth. Hence a result which is sometimes true and sometimes not true is said to be not true.

(ii) **Given:** On Z^+, define $*$ by $a * b = ab$... (*i*)

Let $a \in Z^+ = N$ and $b \in Z^+ = N$; then

$a * b = ab$ is the product of two natural numbers and hence $\in N = Z^+$.

$\therefore$ This operation $*$ is a binary operation, *i.e.,* satisfies **closure law.**

(iii) **Given:** On R_+ define $*$ by $a * b = ab^2$...(*i*)

We know that R_+ denotes the set of positive real numbers including 0.

Let $a \in R_+$ and $b \in R_+ \Rightarrow a \geq 0$ and $b \geq 0$.

$\therefore$ By (*i*), $a * b = ab^2 \geq 0$ *i.e.,* $a * b \in R_+$

$\therefore$ This operation $*$ is a binary operation *i.e.,* satisfies closure law.

(iv) **Given:** On Z^+ define $*$ by $a * b = |a - b|$ ...(*i*)

We know that $a = 2 \in Z^+$ and $b = 2 \in Z^+$

But by (*i*), $a * b = |a - b| = |2 - 2| = 0$ is not a positive integer and hence $\notin Z^+$.

$\therefore$ This operation $*$ is not a binary operation *i.e.,* does not satisfy closure law.

Remark. In the above question if we replace the set Z^+ by Z (the set of positive integers including 0); then by $^{+}$ (i) $a * b = |a - b|$ is either zero or positive integer and hence $\in Z_+$ for all $a, b \in Z_+$ and hence is a binary operation.

(v) **Given:** On Z^+, define $*$ by $a * b = a$ ...(i)

Let $a \in Z^+$ and $b \in Z^+$. Therefore a and b are positive integers. Therefore by (i), $a * b = a$ is also a positive integer and hence $\in Z^+$.

$\therefore$ Operation $*$ defined in (i) is a binary operation *i.e.,* satisfies closure law.

2. For each binary operation $*$ defined below, determine whether $*$ is commutative or associative.

(i) On Z, define $a * b = a - b$

(ii) On Q, define $a * b = ab + 1$

(iii) On Q, define $a * b = \dfrac{ab}{2}$

(iv) On Z^+, define $a * b = 2^{ab}$

(v) \On Z^+, define $a * b = a^b$

(vi) On R − {− 1}, define $a * b = \dfrac{a}{b+1}$

ol. Definition: A binary operation $*$ on a set A is said to be

I **commutative** if $a * b = b * a$ for all $a, b \in$ A

II **Associative** if $(a * b) * c = a * (b * c)$ for all $a, b, c \in$ A.

(i) **Given:** Binary operation on Z, defined as

$a * b = a - b$ ...(*i*)

where $a \in$ Z and $b \in$ Z.

Is $*$ commutative? Interchanging a and b in (*i*),

$$b * a = b - a = -(a - b)$$

From (*i*) and (*ii*), $a * b \neq b * a$ (unless $a = b$)

For example, from (*i*), $2 * 3 = 2 - 3 = -1$

and $3 * 2 = 3 - 2 = 1$ $\therefore \quad 2 * 3 \neq 3 * 2$

$\therefore$ Binary operation $*$ given by (*i*) is not commutative.

Is $*$ associative? Let $a \in$ Z, $b \in$ Z, $c \in$ Z.

$\therefore$ By (*i*), $(a * b) * c = (a - b) * c = (a - b) - c$

$= a - b - c$ ...(*iii*)

Again by (*i*),

$a * (b * c) = a * (b - c) = a - (b - c) = a - b + c$ ...(*iv*)

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

($\because$ Right hand sides are not equal)

For example, $(2 * 3) * 4 = (2 - 3) * 4 = -1 * 4$

$= -1 - 4 = -5$

Again $2 * (3 * 4) = 2 * (3 - 4) = 2 * (-1) = 2 - (-1) = 3$

$\therefore \quad (2 * 3) * 4 \neq 2 * (3 * 4)$

$\therefore$ Binary operation $*$ given by (*i*) is not associative.

Here $*$ is neither commutative nor associative.

(ii) **Given:** Binary operation $*$ on Q defined by

$a * b = ab + 1$ ...(*i*)

for all $a, b \in$ Q

Is $*$ commutative? Interchanging a and b in (*i*),

$b * a = ba + 1 = ab + 1$ ...(*ii*)

[$\because$ We know that $ab = ba$ for all $a, b \in$ Q]

From (*i*) and (*ii*), $a * b = b * a$ for all $a, b \in$ Q

$\therefore$ Binary operation $*$ given by (*i*) is commutative.

Is $*$ associative? Let $a \in$ Q, $b \in$ Q, $c \in$ Q.

By (*i*), $(a * b) * c = (ab + 1) * c = (ab + 1)c + 1$

$= abc + c + 1$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * (bc + 1) = a(bc + 1) + 1$

$= abc + a + 1$ ...(*iv*)

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

[$\because$ Right hand sides of (*iii*) and (*iv*) are not equal]

$\therefore$ $*$ is not associative.

Here $*$ is commutative but not associative.

(iii) **Given:** Binary operation $*$ on Q defined as

$$a * b = \frac{ab}{2} \qquad ...(i)$$

for all $a, b \in Q$.

Is $*$ commutative? Interchanging a and b in (i),

We have $b * a = \frac{ba}{2} = \frac{ab}{2}$ ...(ii)

[$\because$ Ordinary multiplication in Q is commutative]

From (i) and (ii), we have

$$a * b = b * a \text{ for all } a, b \in Q$$

$\therefore$ Binary operation $*$ given by (i) is commutative.

Is $*$ associative? Let $a, b, c \in Q$

By (i), $(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$ ...(iii)

Again by (i), $a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$ ...(iv)

From (iii) and (iv), $(a * b) * c = a * (b * c)$ for all $a, b, c \in Q$

($\because$ Right Hand sides of (iii) and (iv) are equal)

$\therefore$ $*$ is associative.

Here $*$ is both commutative and associative.

(iv) **Given:** Binary operation $*$ on Z^+ defined as

$a * b = 2^{ab}$ ...(i)

for all $a, b \in Z^+$

Is $*$ commutative?

Interchanging a and b in (i), we have

$b * a = 2^{ba} = 2^{ab}$ ...(ii)

From (i) and (ii), we have $a * b = b * a$ for all $a, b \in Z^+$

$\therefore$ $*$ given by (i) is commutative.

Is $*$ associative? Let $a, b, c \in Z^+$.

By (i), $(a * b) * c = 2^{ab} * c = 2^{(2^{ab}).c}$ ...(iii)

Again by (i), $a * (b * c) = a * 2^{bc} = 2^{a\cdot(2^{bc})}$ ...(iv)

From (iii) and (iv) $(a * b) * c \neq a * (b * c)$

$\therefore$ $*$ given by (i) is associative.

$\therefore$ $*$ given by (i), is commutative but not associative.

(v) **Given:** Binary operation $*$ on Z^+ defined as

$a * b = a^b$ ...(i)

for all $a, b \in Z^+$

Is $*$ commutative? Interchanging a and b in (i),

We have $b * a = b^a$ ...(ii)

From (i) and (ii), $a * b \neq b * a$.

$\therefore$ $*$ given by (i) is not commutative.

Is $*$ associative? Let $a, b, c \in Z^+$

By (i), $(a * b) * c = (a^b) * c = (a^b)^c = a^{bc}$ ...(iii)

Again by (2), $a * (b * c) = a * (b^c) = a^{(b^c)}$ ...(iv)

From (iii) and (iv), $(a * b) * c \neq a * (b * c)$

$\therefore$ $*$ is not associative.

$\therefore$ $*$ given by (i) is neither commutative nor associative.

(vi) Since $a * b = \dfrac{a}{b+1}$ (given) ...(i)

Interchanging a and b in (i), $b * a = \dfrac{b}{a+1}$ ...(ii)

$\therefore$ From (i) and (ii), $a * b \neq b * a$

$$\left[e.g.,\ 2 * 3 = \frac{2}{3+1} = \frac{2}{4} = \frac{1}{2} \text{ and } 3 * 2 = \frac{3}{2+1} = \frac{3}{3} = 1\right]$$

$\Rightarrow$ The binary operation '$*$' is not commutative.

Also $(a * b) * c = \dfrac{a}{b+1} * c = \dfrac{\frac{a}{b+1}}{c+1}$ [By (i)]

$$= \frac{a}{(b+1)(c+1)}$$

and $a * (b * c) = a * \dfrac{b}{c+1}$

$$= \frac{a}{\frac{b}{c+1}+1} = \frac{a(c+1)}{b+c+1} \quad \text{[By } (i)]$$

so that $(a * b) * c \neq a * (b * c)$

$$\left[e.g.,\ (2 * 3) * 4 = \frac{2}{3+1} * 4 = \frac{1}{2} * 4 = \frac{1/2}{4+1} = \frac{1}{10} \right.$$

$$\left. \text{and}\quad 2 * (3 * 4) = 2 * \frac{3}{4+1} = 2 * \frac{3}{5} = \frac{2}{3/5+1} = \frac{10}{8} = \frac{5}{4} \right]$$

$\Rightarrow$ The binary operation '$*$' is not associative.

3. Consider the binary operation $\wedge$ on the set {1, 2, 3, 4, 5} defined as $a \wedge b$ = min {a, b}. Write the operation table of the operation $\wedge$.

Sol. Given: Set is {1, 2, 3, 4, 5} = A (say)

Given: Binary operation $\wedge$ on the set A is defined as

$$a \wedge b = \text{min. } \{a, b\} \quad \text{...}(i)$$

Operation (or composition) table of binary operation $\wedge$ given by (i) is being formed below:

$\wedge$	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓
1 →	$1 \wedge 1 = 1$	$1 \wedge 2 = 1$	$1 \wedge 3 = 1$	$1 \wedge 4 = 1$	$1 \wedge 5 = 1$
2 →	$2 \wedge 1 = 1$	$2 \wedge 2 = 2$	$2 \wedge 3 = 2$	$2 \wedge 4 = 2$	$2 \wedge 5 = 2$
3 →	$3 \wedge 1 = 1$	$3 \wedge 2 = 2$	$3 \wedge 3 = 3$	$3 \wedge 4 = 3$	$3 \wedge 5 = 3$
4 →	$4 \wedge 1 = 1$	$4 \wedge 2 = 2$	$4 \wedge 3 = 3$	$4 \wedge 4 = 4$	$4 \wedge 5 = 4$
5 →	$5 \wedge 1 = 1$	$5 \wedge 2 = 2$	$5 \wedge 3 = 3$	$5 \wedge 4 = 4$	$5 \wedge 5 = 5$

(For example) by (i), $4 \wedge 3$ = min. {4, 3} = 3 and by (i),
$2 \wedge 5$ = min. {2, 5} = 2)

Remark. (e =) 5 $\in$ A is the identity element for this binary operation $\wedge$ because the top row headed by the binary operation $\wedge$ coincides with the row headed by 5 in the composition table.
(By definition of identity element $e \in$ A is said to be identity element of binary operation if $a \wedge e = a$ for all $a \in$ A).

4. Consider a binary operation * on the set {1, 2, 3, 4, 5} given by the following multiplication table

(i) Compute (2 * 3) * 4 and 2 * (3 * 4)

(ii) Is * commutative?

(iii) Compute (2 * 3) * (4 * 5).

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Sol. (i) To compute (2 * 3) * 4

Now we know that 2 * 3 is the entry at the intersection of the row headed by 2 and column headed by 3 in the table and this entry is 1.

$\therefore$ (2 * 3) * 4 = 1 * 4 = 1 as explained above.

Again 2 * (3 * 4) = 2 * 1 = 1

$\therefore$ (2 * 3) * 4 = 1 and 2 * (3 * 4) = 1

(ii) **Is $*$ commutative ?**
Let us look at each diagonal entry of the main diagonal starting with $*$. The corresponding entries of the column below any such entry and the adjacent row to the same diagonal entry are same.
$\therefore$ We can say that there is symmetry about this main diagonal.
Better method: The matrix formed by the given table is a **symmetric matrix** and hence $*$ is commutative.
$\therefore$ $*$ is commutative.

(iii) To compute $(2 * 3) * (4 * 5)$
Now $2 * 3$ = Entry in the table at the intersection of row headed by 2 and column headed by $3 = 1$.
Similarly, $4 * 5 = 1$ $\therefore$ $(2 * 3) * (4 * 5) = 1 * 1 = 1$.

5. Let $*'$ be the binary operation on the set {1, 2, 3, 4, 5} defined by $a *' b$ = H.C.F. of a and b. Is the operation $*'$ same as the operation $*$ defined in Exercise 4 above? Justify your answer.

Sol. Given: Binary operation $*'$ on the set {1, 2, 3, 4, 5} is defined as $a *' b$ = H.C.F. of a and b *i.e.,* highest common factor of a and b ...(*i*)

$\therefore$ $2 *' 4$ = H.C.F. of 2 and 4
= (Highest common factor of 2 and 4) = 2
$3 *' 5$ = H.C.F. of 3 and 5
= (Highest common factor of 3 and 5) = 1 etc.

The composition (operation) table for this binary operation $*'$ namely H.C.F. a and b is being given below:

$*'$	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓
1 →	1	1	1	1	1
2 →	$2 *' 1 = 1$	$2 *' 2 = 2$	$2 *' 3 = 1$	$2 *' 4 = 2$	$2 *' 5 = 1$
3 →	1	1	$3 *' 3 = 3$	$3 *' 4 = 1$	1
4 →	1	$4 *' 2 = 2$	1	$4 *' 4 = 4$	1
5 →	1	1	1	1	$5 *' 5 = 5$

Now the corresponding entries in this composition table for this binary operation $*'$ and the composition table given in exercise 4 are same.

$\therefore$ Operation $*'$ of this Exercise 5 and operation $*$ of Exercise 4 defined on the same set {1, 2, 3, 4, 5} are same.

6. Let $*$ be the binary operation on N given by $a * b$ = L.C.M. of a and b. Find

(*i*) $5 * 7$, $20 * 16$ **(*ii*) Is $*$ commutative?**
(iii) Is $*$ associative? **(*iv*) Find the identity of $*$ in N**
(*v*) Which elements of N are invertible for the operation $*$?

Sol. Given: $*$ is a binary operation on N given by

$a * b$ = L.C.M. of a and b ...(*i*)

i.e., least common multiple of a and b.

(i) $\therefore$ $5 * 7$ = L.C.M. of 5 and 7 = $5 \times 7 = 35$

and $20 * 16$ = L.C.M. of 20 and 16

= Least common multiple of 20 and 16. = 80

($\because$ All common multiples of 20 and 16 are 80, 160, 240 etc.)

(ii) **Is $*$ commutative?**

Let $a \in$ N and $b \in$ N

$\therefore$ by (*i*), $a * b$ = L.C.M. of a and b

= L.C.M. of b and a = $b * a$.

$\therefore$ $*$ is commutative.

(iii) **Is $*$ associative?** Let $a \in$ N, $b \in$ N, $c \in$ N

$\therefore$ $(a * b) * c$ = (L.C.M. of a and b) $* c$ [By (*i*)]

= L.C.M. ((L.C.M. of a and b) and c) [By (*i*)]

or $(a * b) * c$ = L.C.M. of a, b, c ...(*ii*)

Again $a * (b * c) = a *$ (L.C.M. of b and c) [By (*i*)]

= L.C.M. (a and (L.C.M. of b and c)) [By (*i*)]

= L.C.M. of a, b and c ...(*iii*)

From (*ii*) and (*iii*), we have $(a * b) * c = a * (b * c)$

$\therefore$ $*$ is associative.

(iv) **To find the identity of $*$ on N.**

Let $a \in$ N. Now $1 \in$ N

By (*i*), $a * 1$ = L.C.M. of a and $1 = a$ for all $a \in$ N

(*i.e.,* $a * e = a$ with $e = 1$)

$\therefore$ $1 \in$ N is the identity element for $*$ on N.

(v) **Which elements of N are invertible for the operation $*$?**

Let $a \in$ N. Let $b \in$ N be inverse of a.

$\therefore$ **By definition of inverse, $a * b = e$**

$\therefore$ By (*i*), L.C.M. of a and b is $e = 1$.

Now this is true only when $a = 1$ and $b = 1$.

($\because$ Only L.C.M. of 1 and 1 is 1)

$\therefore$ $1 \in$N is the only element of N which is invertible.

7. Is $*$ defined on the set {1, 2, 3, 4, 5} by $a * b$ = L.C.M. of a and b a binary operation? Justify your answer.

Sol. Given: Set {1, 2, 3, 4, 5} = A (say)

and operation $*$ is defined on A as $a * b$ = L.C.M. of a and b ...(*i*)

Now $2 \in$ A and $3 \in$ A

But by (*i*), $2 * 3$ = L.C.M. of 2 and 3 is $2 \times 3 = 6 \notin$ A

$3 * 4$ = L.C.M. of 3 and 4 is $3 \times 4 = 12 \notin A$

$\therefore$ Operation $*$ on the set A is not a binary opeartion on the set A.

(*i.e.,* operation $*$ given by (*i*) does not satisfy **closure law**).

Remark. For still better understanding of the concept, the reader is suggested to construct the operation (*i.e.,* composition) table for the operation given by (*i*) and then observe that entries 6, 10, 12, 15, 20 of the composition table don't belong to A and hence conclude that operation $*$ is not a binary operation.

8. Let $*$ be the binary operation on N defined by $a * b$ = H.C.F. of a and b. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N?

Sol. $*$ is a binary operation on N defined as:

$a * b$ = H.C.F. of a and b ...(*i*)

Is $*$ commutative?

Reproduce Exercise 6(*ii*) replacing the phrase "L.C.M." by "H.C.F."

Is $*$ associative?

Reproduce Exercise 6(*iii*) replacing the phrase "L.C.M." by "H.C.F."

Does there exists identity for this binary operation on N?

We know that there does not exist any natural number e such that H.C.F. of a and e is a for all $a \in$ N. It may be noted that $e \neq 1$ even because $a * 1$ = H.C.F. of a and 1 is 1 and $\neq a$.

(For identity $a * e = a$ and $\neq e$).

9. Let $*$ be a binary operation on the set Q of rational numbers as follows:

(*i*) $a * b = a - b$ **(*ii*) $a * b = a^2 + b^2$**

(iii) $a * b = a + ab$ **(*iv*) $a * b = (a - b)^2$**

(*v*) $a * b = \frac{ab}{4}$ **(*vi*) $a * b = ab^2$.**

Find which of the binary operations are commutative and which are associative.

Sol. (*i*) Reproduce the solution of Exercise 2(*i*) replacing Z by Q.

(ii) The given binary operation $*$ is $a * b = a^2 + b^2$ ...(*i*)

for all $a, b \in$ Q.

Is $*$ commutative?

Interchanging a and b in (*i*), we have

$$b * a = b^2 + a^2 = a^2 + b^2 \quad ...(ii)$$

[$\because$ Addition is commutative in Q]

From (*i*) and (*ii*), we have $a * b = b * a$ for all $a, b \in$ Q.

$\therefore$ $*$ is commutative on Q.

Is $*$ associative? Let $a, b, c \in$ Q.

By (*i*), $(a * b) * c = (a^2 + b^2) * c = (a^2 + b^2)^2 + c^2$

$= a^4 + b^4 + 2a^2b^2 + c^2$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * (b^2 + c^2) = a^2 + (b^2 + c^2)^2$

$= a^2 + b^4 + c^4 + 2b^2c^2$ ...(*iv*)

From (*iii*) and (*iv*) $(a * b) * c \neq a * (b * c)$

$\therefore$ Binary operation $*$ is commutative but not associative.

(iii) The given binary operation $*$ on Q is $a * b = a + ab$ ...(*i*)
for all $a, b \in$ Q

Is $*$ commutative?

Interchanging a and b in (*i*)

We have $b * a = b + ba = b + ab$ ...(*ii*)

From (*i*) and (*ii*), $a * b \neq b * a$

$\therefore$ $*$ is not commutative.

Is $*$ associative?

Let $a, b, c \in$ Q

From (*i*), $(a * b) * c = (a + ab) * c = a + ab + (a + ab)\,c$

$= a + ab + ac + abc$ ...(*iii*)

Again from (*i*), $a * (b * c) = a * (b + bc)$

$= a + a(b + bc) = a + ab + abc$ ...(*iv*)

From (*iii*) and (*iv*) $(a * b) * c \neq a * (b * c)$

$\therefore$ $*$ is not associative.

$\therefore$ Binary operation $*$ is neither commutative nor associative.

(iv) The given binary operation $*$ on Q is

$a * b = (a - b)^2$ ...(*i*)

for all $a, b \in$ Q

Interchanging a and b in (*i*), we have

$b * a = (b - a)^2 = (-(a - b))^2 = (a - b)^2 = a * b$ [By (*i*)]

$\therefore$ Binary operation $*$ is commutative.

Is $*$ associative?

Let $a, b, c \in$ Q

By (*i*), $(a * b) * c = (a - b)^2 * c = ((a - b)^2 - c)^2$

$= (a^2 + b^2 - 2ab - c)^2$ ...(*ii*)

Again by (*i*),

$a * (b * c) = a * (b - c)^2 = (a - (b - c)^2)^2$

$= (a - (b^2 + c^2 - 2bc))^2 = (a - b^2 - c^2 + 2bc)^2$

$= (-(b^2 + c^2 - 2bc - a))^2$

$= (b^2 + c^2 - 2bc - a)^2$ ...(*iii*)

From (*ii*) and (*iii*), $(a * b) * c \neq a * (b * c)$

$\therefore$ Binary operation $*$ is not associative.

$\therefore$ Binary operation $*$ is neither commutative nor associative.

(v) **Given:** Binary operation $*$ on Q defined as $a * b = \frac{ab}{4}$ for all $a, b \in$ Q ...(i)

Is $*$ commutative?

Interchanging a and b in (i), $b * a = \frac{ba}{4} = \frac{ab}{4}$ ...(ii)

(Multiplication is commutative in Q)

From (i) and (ii), we have $a * b = b * a$ for all $a, b \in$ Q.

$\therefore$ $*$ is commutative on Q.

Is $*$ associative on Q?

Let $a \in$ Q, $b \in$ Q, $c \in$ Q.

By (i), $(a * b) * c = \left(\frac{ab}{4}\right) * c = \frac{\left(\frac{ab}{4}\right)c}{4} = \frac{abc}{16}$ ...(iii)

Again by (i), $a * (b * c) = a * \frac{bc}{4} = \frac{a\left(\frac{bc}{4}\right)}{4} = \frac{abc}{16}$ ...(iv)

From (*iii*) and (*iv*), we have $(a * b) * c = a * (b * c)$ for all $a, b, c \in Q$.

$\therefore$ $*$ is associative. $\therefore$ is commutative as well as associative.

(vi) **Given:** Binary operation $*$ on Q defined as

$$a * b = ab^2 \quad ...(i)$$

for all, $a, b \in Q$

Is $*$ commutative?

Interchanging a and b in (*i*), we have

$$b * a = ba^2 \quad ...(ii)$$

From (*i*) and (*ii*), we have $a * b \neq b * a$

$\therefore$ $*$ is not commutative.

Is $*$ associative? Let $a, b, c \in Q$.

By (*i*), $(a * b) * c = (ab^2) * c\ (ab^2)\ c^2 = ab^2c^2$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * (bc^2) = a\ (bc^2)^2$

$$= ab^2c^4 \quad ...(iv)$$

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

$\therefore$ $*$ is not associative.

$\therefore$ $*$ is neither commutative nor associative.

10. Find which of the operations given above has identity.

Sol. (*i*) **Existence of identity for $*$.**

The binary operation on the set Q of rational numbers is

defined as $a * b = a - b$.. (*i*)

If possible, let $e \in Q$ be the identity for $*$.

$\therefore$ $a * e = e * a\ (= a)$ for all $a \in Q$.

$\therefore$ By (*i*), $a - e = e - a \Rightarrow -2e = -2a \Rightarrow e = a \in Q$

$\therefore$ e has infinite values. But this is impossible [......Identity e for a binary operation is unique]

$\therefore$ Identity for this $*$ does not exist.

Even $0 \in Q$ is not the identity for binary operation $*$ given by (*i*) as $a * 0\ (= a - 0 = a \neq 0 * a\ (= 0 - a = -\ a)$ for all $a \in Q$

(ii) **Existence of identity for $*$.**

Identity element $e \in Q$ does not exist for $*$ given by (*i*), as

$$a * e = a^2 + e^2 \text{ can never be equal to } a.$$

(iii) **Existence of Identity for $*$**

If possible, let $e \in Q$ be the identity of binary operation $*$ given by (*i*).

$\therefore$ $a * e = e * a\ (= a)$ $\therefore$ By (*i*), $a + ae = e + ea$

$\Rightarrow a = e \Rightarrow e = a \in Q$

$\Rightarrow$ e has infinite values $a \in Q$

But this is impossible because we know that e is unique.

$\therefore$ e does not exist for this binary operation $*$.

(iv) **Existence of Identity for $*$**

If possible let $e \in Q$ be the identity for $*$ given by (i).

Therefore, $a * e = a$ for all $a \in Q$.

$\therefore$ By (i), $(a - e)^2 = a$ which is impossible to hold true for any $e \in Q$.

It is not true even for $e = 0$ because for $e = 0$, the above equation becomes $a^2 = a$ which is not true for every $a \in Q$.

$\therefore$ Identity element does not exist for this binary operation $*$.

(v) **Does there exist identity element for this binary operation $*$?**

If possible, let $e \in Q$, be the identity element for $*$.

$\therefore$ $a * e = a \Rightarrow$ By (i), $\dfrac{ae}{4} = a$

Cross-multiplying, $ae = 4a$

Dividing by a (if $a \neq 0$ *i.e.*, if Q is replaced by Q – {0}); then $e = 4 \in Q - \{0\}$ is the identity element of this $*$ on Q – {0}.

$\therefore$ For this binary operation $*$, e does not exist for Q but e exists for Q – {0} and $e = 4$.

(vi) **Does there exist identity element for this binary operation $*$?**

If possible, let $e \in Q$ be the identity element for this binary operation $*$.

$\therefore \quad a * e = a \quad \Rightarrow \quad$ by (i), $ae^2 = a$

Dividing by a ($\neq 0$ if $a \in Q - \{0\}$), $e^2 = 1$

$\therefore \quad e = \pm 1 \in Q - \{0\}$

Even now e is not unique (because e has two values 1 and – 1)

$\therefore$ e does not exist for binary operation $*$ on Q nor exists for binary operation $*$ if Q is replaced by Q – {0}.

11. Let A = N × N and $*$ be the binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$.

Show that $*$ is commutative and associative. Find the identity element for $*$ on A, if any.

Sol. For all $(a, b) \in A = N \times N$, we are given that

$$(a, b) * (c, d) = (a + c, b + d) \quad \ldots(i)$$
$$= (c + a, d + b)$$
[$\because$ Addition is commutative on N]
$$= (c, d) * (a, b) \quad \text{[By } (i)]$$

$\Rightarrow$ $*$ is commutative.

For all $(a, b), (c, d), (e, f) \in A = N \times N$, we have

$$[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f) \quad \text{[By } (i)]$$
$$= ((a + c) + e, (b + d) + f) \quad \text{[By } (i)]$$
$$= (a + (c + e), b + (d + f))$$
[$\because$ Addition is associative on N]
$$= (a, b) * (c + e, d + f) \quad \text{[By } (i)]$$
$$= (a, b) * [(c, d) * (e, f)] \quad \text{(By } (i))$$

$\Rightarrow$ $*$ is associative.

Now to find the identity element for $*$ on A, if any

Now suppose (x, y) is the identity element in A = N × N: Then

$$(a, b) * (x, y) = (a, b), \forall (a, b) \in A \ (a * e = a)$$

$\Rightarrow \quad (a + x, b + y) = (a, b)$ [By (i)]

$\Rightarrow \quad a + x = a$ and $b + y = b$

$\Rightarrow \quad x = 0$ and $y = 0$

But $0 \notin N$, therefore $(0, 0) \notin A = N \times N$

Hence $*$ has no identity element.

12. State whether the following statements are true or false. Justify.

(i) For any arbitrary binary operation $*$ on a set N,
$a * a = a \ \forall \ a \in N$

(ii) If $*$ is commutative binary operation on N, then
$a * (b * c) = (c * b) * a$.

Sol. (i) **False.** Here given $a * a = a \ \forall \ a \in N$ ...(i)

Because we know that a binary operation $*$ on a set N is a function from N × N → N and by definition of function, Image of every element (a, b) of domain N × N must be possible.

But by given rule (i),

Image of (a, b) $(b \neq a)$ is undefined.

(ii) **True.** Because L.H.S. $= a * (b * c) = (b * c) * a$

[$\because$ $*$ is given to be commutative binary operation on N and $b * c \in$ N, $a \in$ N]

$= (c * b) * a$

[$\because$ $*$ is given to be commutative binary operation]

= R.H.S.

13. Consider a binary operation $*$ on N defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is $*$ both associative and commutative?

(B) Is $*$ commutative but not associative?

(C) Is $*$ associative but not commutative?

(D) Is $*$ neither commutative nor associative?

Sol. Binary operation $*$ on N is defined as $a * b = a^3 + b^3$ ...(i)

for all $a, b \in$ N

Is $*$ commutative?

Interchanging a and b in (i),

$$b * a = b^3 + a^3 = a^3 + b^3 \quad ...(ii)$$

($\because$ Usual addition is commutative in N)

From (i) and (ii), $a * b = b * a$ $\therefore$ $*$ is commutative.

Is $*$ associative?

Let $a, b, c \in$ N

By (i), $(a * b) * c = (a^3 + b^3) * c = (a^3 + b^3)^3 + c^3$ ...(iii)

Again $a * (b * c) = a * (b^3 + c^3) = a^3 + (b^3 + c^3)^3$ ...(iv)

From (iii) and (iv), $(a * b) * c \neq a * (b * c)$

$\therefore$ Binary operation $*$ is not associative.

$\therefore$ Binary operation $*$ is commutative but not associative.

$\therefore$ Option (B) is the correct answer

MISCELLANEOUS EXERCISE

1. **Let f : R → R be defined by $f(x) = 10x + 7$. Find the function g : R → R such that $gof = fog = I_R$.**

Sol. Given: $f : R \to R$ defined by $f(x) = 10x + 7$ ...(*i*)

We are to find a function $g : R \to R$ such that $gof = fog = I_R$

$\because$ It is given that $gof = fog = I_R$, therefore by definition of inverse of a function, f^{-1} exists and $g = f^{-1}$ ...(*ii*)

$\therefore$ To find function $g \Rightarrow$ To find f^{-1}.

Because f^{-1} exists, therefore f is one-one onto function.

$\therefore$ $y(= f(x)) = 10x + 7$ ...(*iii*)

Let us find *x* in terms of *y*.

From (*iii*), $y - 7 = 10x$

$\Rightarrow \quad x = \dfrac{y-7}{10} \quad \Rightarrow \quad f^{-1}(y) = \dfrac{y-7}{10} \quad \Big| \because \quad y = f(x) \Rightarrow x = f^{-1}(y)$

$\Rightarrow \quad g(y) = \dfrac{y-7}{10}$. $\quad$ ($\because$ By (*ii*), $f^{-1} = g$)

2. Let f : W → W be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Sol. We know that W, the set of all whole numbers = N ∪ {0} where N is the set of all natural numbers.

For one-one function: Let $n_1, n_2 \in$ W with $f(n_1) = f(n_2)$

If n_1 is odd and n_2 is even, then by the given rule of the function f, we will have $n_1 - 1 = n_2 + 1$, *i.e.,* $n_1 - n_2 = 2$ which is impossible since the difference of an odd and an even number is always odd. Similarly, n_1 is even and n_2 is odd is ruled out. Therefore, both n_1 and n_2 must be either odd or even.

Suppose n_1 and n_2 both are odd, then by the given rule of the function f, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 - 1$

$\Rightarrow \quad n_1 = n_2$. Similarly, if n_1 and n_2 both are even, then $f(n_1) = f(n_2)$

$\Rightarrow \quad n_1 + 1 = n_2 + 1 \quad \Rightarrow \quad n_1 = n_2$. $\quad \therefore$ f is one-one.

Now let us prove that f is onto

According to given,

$f(n) = n - 1$ if n is odd $\Rightarrow f(n)$ is even ($\because$ Odd − 1 = Even) ...(*i*)

and $f(n) = n + 1$ if n is even

$\Rightarrow \quad f(n)$ is odd $\quad$ ($\because$ Even + 1 = Odd) ...(*ii*)

Let $y \in$ Co-domain W

Case I. y is Even

$\therefore$ From (*i*), $y = f(n) = n - 1 \quad \therefore n = y + 1$ is odd ...(*iii*)

Case II. y is odd

$\therefore$ From (*ii*), $y = f(n) = n + 1 \quad \therefore n = y - 1$ is even ...(*iv*)

$\therefore$ From (*iii*), for every even $y \in$ co-domain W,

there exsts odd $n = y + 1$ such that $f(n) = y$.

and from (*iv*), for every odd $y \in$ co-domain W,

there exists even $n = y - 1$ such that $f(n) = y$. $\quad \therefore$ f is onto.

Thus, f is one-one and onto and therefore invertible.

$\therefore \quad f(n) = y \quad \Rightarrow \quad f^{-1}(y) = n$

$\therefore$ From (*iv*) and (*iii*), we have

$$f^{-1}(y)\ (= n) = \begin{cases} y-1 & \text{if } y \text{ is odd} \quad \ldots(v) \\ y+1 & \text{if } y \text{ is even} \quad \ldots(vi) \end{cases}$$

From (*i*), (*ii*), (*v*) and (*vi*); $f = f^{-1}$

Note. $0 \in W$ is even.

3. Let $f : R \to R$ be defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Sol. $f : R \to R$ is defined by $f(x) = x^2 - 3x + 2$ ...(*i*)

$\therefore \quad f(f(x)) = f(x^2 - 3x + 2)$ ...[By (*i*)]

Changing x to $x^2 - 3x + 2$ in (*i*)

$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$

$(\because \; (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac)$

$= x^4 - 6x^3 + 10x^2 - 3x.$

4. Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in R$ is one-one and onto function.

Sol. Given: $f : R \to \{x \in R : -1 < x < 1\}$

given by $f(x) = \dfrac{x}{1+|x|}$ ...(*i*)

To prove: *f* is one-one

Case I. $x \geq 0$

$\therefore \quad |x| = x$ and therefore from (*i*), $f(x) = \dfrac{x}{1+x}$ ...(*ii*)

Let $x_1 \in R$, $x_2 \in R$ (such that $x_1 \geq 0$, $x_2 \geq 0$) and $f(x_1) = f(x_2)$

$\therefore$ From (*ii*), $\dfrac{x_1}{1+x_1} = \dfrac{x_2}{1+x_2}$

Cross-multiplying, $x_1(1 + x_2) = x_2(1 + x_1)$

$\Rightarrow \quad x_1 + x_1x_2 = x_2 + x_1x_2 \quad \Rightarrow \quad x_1 = x_2.$

Case II. $x < 0$

$\therefore \; |x| = -x$ and therefore from (*i*), $f(x) = \dfrac{x}{1-x}$ ...(*iii*)

Let $x_1 \in R$, $x_2 \in R$ (such that $x_1 < 0$, $x_2 < 0$) and $f(x_1) = f(x_2)$

$\therefore$ From (*iii*), $\dfrac{x_1}{1-x_1} = \dfrac{x_2}{1-x_2}$

Cross-multiplying, $x_1(1 - x_2) = x_2(1 - x_1)$

or $\quad x_1 - x_1x_2 = x_2 - x_1x_2$ or $x_1 = x_2$

$\therefore$ In both the cases I and II,

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$ $\therefore$ *f* is one-one.

To prove: *f* is onto.

According to given,

Co-domain $= \{x : -1 < x < 1\}$

i.e., $= \{y : -1 < y < 1\}$ = open interval $(-1, 1)$ ($\because$ Elements of co-domain are generally denoted by y) **Let us find range $f(x)$**

From eqn. (*ii*), for $x \geq 0$, $f(x) = \dfrac{x}{1+x}$

We know that for $x \geq 0$, $x < 1 + x$

Dividing by $1 + x$, $\dfrac{x}{1+x} < 1$ *i.e.,* $f(x) < 1$ ...(*iv*)

From eqn. (*iii*) for $x < 0$, $f(x) = \dfrac{x}{1-x}$

We know that for $x < 0$, $x > x - 1$

($\because$ $x - 1$ is more negative than x)

$\Rightarrow$ $x > -1\,(1 - x)$

Dividing both sides by $(1 - x)$ (> 0), $\dfrac{x}{1-x} > -1$

i.e., $f(x) > -1$ *i.e.*, $-1 < f(x)$ ...(*v*)

From (*v*) and (*iv*) $-1 < f(x) < 1$

i.e., Range set = open interval $(-1, 1)$ = co domain (given)

$\therefore$ f is onto. $\therefore$ f is one-one and onto.

5. Show that the function $f : R \to R$ given by $f(x) = x^3$ is injective.

Sol. Function $f : R \to R$ given by $f(x) = x^3$ ...(*i*)

Let $x_1, x_2 \in$ domain R such that $f(x_1) = f(x_2)$

$\therefore$ By (*i*), $x_1^3 = x_2^3$ $\Rightarrow$ $x_1 = x_2$

$\therefore$ f is injective (function) *i.e.*, one-one function.

6. Give examples of two functions $f : N \to Z$ and $g : Z \to Z$ such that gof is injective but g is not injective.

Sol. Let us define $f : N \to Z$ as $f(x) = x$ ...(*i*)

and $g : Z \to Z$ as $g(x) = |x|$ ...(*ii*)

To prove: g is not injective *i.e.*, g is not one-one.

Now $x_1 = -1 \in Z$, $x_2 = 1 \in Z$

From (*ii*), $g(x_1) = g(-1) = |-1| = 1$

and $g(x_2) = g(1) = |1| = 1$

Now $g(x_1) = g(x_2)$ $(= 1)$ but $x_1 (= -1) \neq x_2 (= 1)$

$\therefore$ g is not injective.

Let us find the function gof

$\because$ $f : N \to Z$ and $g : Z \to Z$, therefore $gof : N \to Z$

($\because$ Domain of gof by def. is always same as domain of f)

and $(gof)(x) = g(f(x)) = g(x)$ [By (*i*)]

$= |x|$ [By (*ii*)]

i.e., $(gof)x = |x|$ for all $x \in$ domain N

$= x$ ...(*iii*)

($\because$ $x \in N \Rightarrow x \geq 1 \Rightarrow x > 0$ and hence $|x| = x$)

To prove: gof is one-one.

Let $x_1, x_2 \in N$ (domain of gof) such that

$(gof)x_1 = (gof)\,x_2$ $\Rightarrow$ By (*iii*), $x_1 = x_2$

$\therefore$ gof is one-one (*i.e.*, injective)

$\therefore$ For the above example, gof is injective but g is not injective.

Remark. A second example for the above question is:

take $f(x) = 2x$ and $g(x) = x^2$.

7. Give examples of two functions $f : N \to N$ and $g : N \to N$ such that gof is onto but f is not onto.

Sol. Let $f : \mathrm{N} \to \mathrm{N}$ be defined by $f(x) = 2x$ ...(*i*)

$\therefore$ Range set $= \{f(x) = 2x : x \in \mathrm{N}\}$

$= \{2 \times 1, 2 \times 2, 2 \times 3, ...\} = \{2, 4, 6, 8, 10, ...\}$

$=$ Set of even natural numbers $\neq$ co-domain N.

$\therefore$ f is not onto.

Let $g : \mathrm{N} \to \mathrm{N}$ be defined by $g(x) = \frac{1}{2}x$ ...(*ii*)

$\therefore$ gof is a function from $\mathrm{N} \to \mathrm{N}$ and is defined as

$$(gof)\,x = g(f(x)) = g(2x) \quad = \frac{1}{2}(2x) \qquad \text{[By } (i)]$$

$$= x$$

$\therefore$ $(gof) : \mathrm{N} \to \mathrm{N}$ and $(gof)x = x$

$\therefore$ $(gof)(1) = 1$, $(gof)(2) = 2$ etc.

i.e., Range of gof is N and equals to co-domain N.

$\therefore$ gof is onto but f is not onto (proved above).

8. **Given a non-empty set X; consider P(X) which is the set of all subsets of X.**

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if A $\subset$ B. Is R an equivalence relation on P(X)? Justify your answer.

Sol. Given: The set P(X) which is the set of all subsets of X.

Given: For subsets A, B in P(X); ARB if and only if

$\mathrm{A} \subset \mathrm{B}$ ...(*i*)

Is R reflexive? Let $\mathrm{A} \in \mathrm{P(X)}$.

Putting B = A in (*i*), we have $\mathrm{A} \subset \mathrm{A}$

which is true. ($\because$ Every set is a subset of itself)

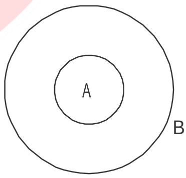
$\therefore$ By (*i*), ARA and hence R is reflexive.

Is R symmetric? Let $\mathrm{A} \in \mathrm{P(X)}$ and $\mathrm{B} \in \mathrm{P(X)}$ and ARB

$\therefore$ By (*i*), $\mathrm{A} \subset \mathrm{B}$

$\therefore$ B is not a subset of A

(See the adjoining figure)



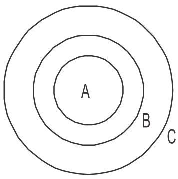
$\therefore$ B is not related to A.

Hence R is not symmetric.

Is R transitive? Let $\mathrm{A} \in \mathrm{P(X)}$, $\mathrm{B} \in \mathrm{P(X)}$ and $\mathrm{C} \in \mathrm{P(X)}$ such that ARB and BRC.

$\therefore$ By (*i*), $\mathrm{A} \subset \mathrm{B}$ and $\mathrm{B} \subset \mathrm{C}$

$\therefore$ $\mathrm{A} \subset \mathrm{C}$ $\therefore$ By (*i*), ARC. $\therefore$ R is transitive.



$\therefore$ R is reflexive and transitive but not symmetric. Hence R is not an equivalence relation on the set P(X).

9. **Given a non-empty set X, consider the binary operation $*$: $\mathrm{P(X)} \times \mathrm{P(X)} \to \mathrm{P(X)}$ given by $\mathrm{A} * \mathrm{B} = \mathrm{A} \cap \mathrm{B}\ \forall\ \mathrm{A}, \mathrm{B}$ in P(X), where P(X) is the power set of X. Show that X is the**

identity element for this operation and X is the only invertible element in P(X) with respect to the operation $*$.

Sol. We know that P(X), the power set of X is the set of all subsets of the set X.

Given: The binary operation $*$:

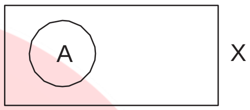
$P(X) \times P(X) \to P(X)$ defined as $A * B = A \cap B$ ...(*i*)

for all A, B in P(X).

To show that the set $X \in P(X)$ is the identity element for the binary operation $*$ defined in (*i*)

Let $A \in P(X)$ $(\Rightarrow A \subset X)$

By (*i*) $A * X = A \cap X = A$ (can be seen from the adjoining figure)



similarly $X * A = X \cap A = A$

$\therefore$ $A * X = A = X * A$ $\forall A \in P(X)$

$\therefore$ set X is the identity element for the binary operation $*$. (*ii*)

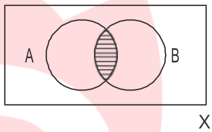
$(a * e = a = e * a)$

To find invertible elements in P(X) w.r.t. binary operation $*$.

Let $A \in P(X)$ be an invertible element of P(X).

Therefore there exists $B \in P(X)$.

(By def. of inverse)



such that $A * B = E$

$\therefore$ By (*i*), $A \cap B = X$ [by (*ii*)]

and this equation can hold true only when A = X and also B = X. ($\because$ Only $X \cap X = X$)

$\therefore$ A = X is the only invertible element of P(X) (having B = X as its inverse).

10. Find the number of all onto functions from the set {1, 2, 3, ..., *n*} to itself.

Sol. The given set is $\{1, 2, 3, ..., n\} = A$ (say)

This set A is a finite set having only n elements.

We know that every **onto** function from a **finite** set $A \to A$ is **one-one** and conversely every one-one function from finite set $A \to A$ is also **onto.**

$\therefore$ Number of **onto** functions f from A to A here

= Number of **one-one** functions f from A to A ...(*i*)

Since f is one-one, therefore image of $1 \in A$ can be any one of the n elements of A.

Now image of $2 \in A$ can be any one of the remaining $(n - 1)$ elements of A ($\because$ f is one-one $\Rightarrow$ No two elements have the same image *i.e.,* repetition is not allowed.)

Similarly, image of $3 \in A$ can be any one of the remaining $(n - 2)$ elements of A and so on.

$\therefore$ Total number of **onto** functions from $A \to A$

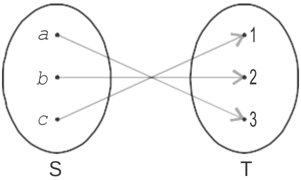
(*i.e.,* one-one functions from $A \to A$). [By (*i*)]

$= n(n - 1)(n - 2) \ldots 1$

$= n!$. [By Multiplication Rule of Principle of counting]

11. Let S = $\{a, b, c\}$ and T = $\{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T, if it exists.
(i) F = $\{(a, 3), (b, 2), (c, 1)\}$ (*ii*) F = $\{(a, 2), (b, 1), (c, 1)\}$.

Sol. (*i*) **Given:** S = $\{a, b, c\}$ and T = $\{1, 2, 3\}$.



Given: Function F : S → T is

$$F = \{(a, 3), (b, 2), (c, 1)\}$$

$\Rightarrow$ $F(a) = 3$, $F(b) = 2$, $F(c) = 1$.

This function F is one-one because distinct element a, b, c have distinct images 3, 2, 1.

F is onto because range set

$= \{3, 2, 1\} = \{1, 2, 3\}$ = co-domain T.

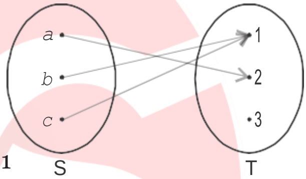
$\therefore$ F is one-one and onto function and hence F^{-1} exists and is given by

$$F^{-1}(3) = a,\ F^{-1}(2) = b, \text{ and } F^{-1}(1) = c$$

$\therefore$ Function $F^{-1} = \{(3, a), (2, b), (1, c)\}$

(ii) **Given:** S = $\{a, b, c\}$ and

T = $\{1, 2, 3\}$



Given: Function

$$F = \{(a, 2), (b, 1), (c, 1)\}$$

$\Rightarrow$ $F(a) = 2$, $F(b) = 1$ and $F(c) = 1$

$\therefore$ F is not one-one because $F(b) = F(c)(= 1)$ but $b \neq c$ *i.e.,* two elements b and c of the domain have same image 1.

$\therefore$ F is not onto because range set of F is $\{2, 1\}$ $\neq$ co-domain

T = $\{1, 2, 3\}$. $\therefore$ F^{-1} does not exist.

[$\because$ F^{-1} exists iff F is one-one and onto]

12. Consider the binary operations $*$: R × R → R and o : R × R → R defined as $a * b = |a - b|$ and $a \circ b = a, \forall\, a, b \in$ R. Show that $*$ is commutative but not associative, o is associative but not commutative. Further, show that $\forall\, a, b, c \in$ R, $a * (b \circ c) = (a * b) \circ (a * c)$. [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

Sol. Given: Binary operation $*$: R × R → R is defined as

$$a * b = |a - b| \quad ...(i)$$

for all $a, b \in$ R.

Is $*$ commutative?

Interchanging a and b in (*i*), we have

$$b * a = |b - a| = |-(a - b)| = |a - b| \quad ...(ii)$$

[$\because$ $|-t| = |t|\ \forall\, t \in$ R]

From (*i*) and (*ii*), we have $a * b = b * a\ \forall\, a, b \in$ R.

$\therefore$ $*$ is commutative.

Is $*$ associative? Let $a, b, c \in R$.

By (*i*), $(a * b) * c = |a - b| * c = ||a - b| - c|$ ...(*iii*)

Again by (*i*), $a * (b * c) = a * |b - c| = |a - |b - c||$ ...(*iv*)

From (*iii*) and (*iv*), $(a * b) * c \neq a * (b * c)$

For example, $(2 * 3) * 4 = |2 - 3| * 4 = 1 * 4 = |1 - 4| = 3$

Again $2 * (3 * 4) = 2 * |3 - 4| = 2 * 1 = |2 - 1| = 1$

$\therefore$ $(2 * 3) * 4 \neq 2 * (3 * 4)$

$\therefore$ Binary operation $*$ is not associative.

Also given: o is a binary operation from $R \times R \to R$ defined as

$a \text{ o } b = a \quad \forall\, a, b \in R$ ...(*v*)

Is o commutative? Interchanging a and b in (*v*), we have $b \text{ o } a = b$ ...(*vi*)

From (*v*) and (*vi*) $a \text{ o } b \neq b \text{ o } a$

$\therefore$ o is not commutative.

Is o associative? Let $a, b, c \in R$.

By (*v*), $(a \text{ o } b) \text{ o } c = a \text{ o } c = a$ ...(*vii*)

Again by (*v*), $a \text{ o } (b \text{ o } c) = a \text{ o } b = a$ ...(*viii*)

From (*vii*) and (*viii*) $(a \text{ o } b) \text{ o } c = a \text{ o } (b \text{ o } c)$

$\therefore$ o is associative.

Now we are to prove that $a * (b \text{ o } c) = (a * b) \text{ o } (a * c)$

L.H.S. $= a * (b \text{ o } c) = a * b$ [By (*v*)]

$= |a - b|$ ...(*ix*) [By (*i*)]

Again R.H.S. $= (a * b) \text{ o } (a * c) = |a - b| \text{ o } |a - c|$ [By (*i*)]

$= |a - b|$ ...(*x*) [By (*v*)]

From (*ix*) and (*x*), we have L.H.S. = R.H.S.

i.e., $a * (b \text{ o } c) = (a * b) \text{ o } (a * c)$ ...(*xi*)

$\therefore$ We can say that the operation $*$ distributes itself over the operation o.

Now we are to examine if binary operation o distributes over binary operation $*$.

i.e., Is eqn. (*xi*) true on interchanging $*$ and o.

i.e., if $a \text{ o } (b * c) = (a \text{ o } b) * (a \text{ o } c)$ ...(*xii*)

L.H.S. $= a \text{ o } (b * c) = a \text{ o } |b - c|$ [By (*i*)]

$= a$ [By (*v*)]

Again, R.H.S. $= (a \text{ o } b) * (a \text{ o } c) = a * a$ [By (*v*)]

$= |a - a| = 0$ [By (*i*)]

$\therefore$ L.H.S. $\neq$ R.H.S.

$\therefore$ Eqn. (*xii*) is not true. *i.e.,* o does not distribute over $*$.

13. Given a non-empty set X, let $* : P(X) \times P(X) \to P(X)$ be defined as $A * B = (A - B) \cup (B - A)$, $\forall\, A, B \in P(X)$. Show that the empty set ϕ is the identity for the operation $*$ and all the elements A of P(X) are invertible with $A^{-1} = A$.

Sol. $A * B = (A - B) \cup (B - A) \;\forall\; A, B \in P(X)$ ...(*i*) (given)

Replacing B by ϕ in (*i*),

$$A * \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$$

$\Rightarrow \quad A * \phi = A \;\forall\; A \in P(X)$ $\quad \therefore \quad \phi$ is the identity for $*$

Again replacing B by A in eqn. (*i*), we have

$$A * A = (A - A) \cup (A - A) = \phi \cup \phi = \phi$$

$\Rightarrow \quad A * A = \phi$ $\quad \Rightarrow$ A is invertible and $A^{-1} = A$.

Remark. We know that $(A - B) \cup (B - A)$ is called **symmetric difference** of sets A and B is denoted by $A \Delta B$.

14. **Define a binary operation $*$ on the set {0, 1, 2, 3, 4, 5} as**

$$a * b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a.

Sol. Given: set {0, 1, 2, 3, 4, 5} = A (say)

Given: binary operation $*$ defined on the set A is defined as

$$a * b = \begin{cases} a + b, & \text{if } a + b < 6 \quad ...(i) \\ a + b - 6 & \text{if } a + b \geq 6 \quad ...(ii) \end{cases}$$

The composition (operation) Table for this binary operation $*$ defined by (*i*) and (*ii*) is being given below:

$\overset{*}{\rightarrow}$	0	1	2	3	4	5
0 $\rightarrow$	0 + 0 = 0 by (*i*)	0 + 1 = 1 by (*i*)	0 + 2 = 2 by (*i*)	0 + 3 = 3 by (*i*)	0 + 4 = 4 by (*i*)	0 + 5 = 5 by (*i*)
1	1 + 0 = 1 by (*i*)	1 + 1 = 2 by (*i*)	1 + 2 = 3 by (*i*)	1 + 3 = 4 by (*i*)	1 + 4 = 5 by (*i*)	1 + 5 − 6 = 0 by (*ii*)
2	2 + 0 = 2 by (*i*)	2 + 1 = 3 by (*i*)	2 + 2 = 4 by (*i*)	2 + 3 = 5 by (*i*)	2 + 4 − 6 = 0 by (*ii*)	2 + 5 − 6 = 1 by (*ii*)
3	3 + 0 = 3 by (*i*)	3 + 1 = 4 by (*i*)	3 + 2 = 5 by (*i*)	3 + 3 − 6 = 0 by (*ii*)	3 + 4 − 6 = 1 by (*ii*)	3 + 5 − 6 = 2 by (*ii*)
4	4 + 0 = 4 by (*i*)	4 * 1 = 4 + 1 = 5	4 * 2 = 4 + 2−6 = 0 by (*ii*) (m4+ 2=6)	4 * 3 = 4 + 3−6 =1 by (*ii*)	4 * 4 4 + 4 − 6 = 2 by (*ii*)	4 * 5 = 4 + 5 − 6 = 3 by (*ii*)
5	5 * 0 = 5 + 0 = 5	5 * 1 = 5 + 1 − 6 = 0 by (*ii*)	5 * 2 = 5 + 2−6 = 1 by (*ii*)	5 * 3 = 5 + 3 − 6 = 2	5 * 4 = 5 + 4 − 6 = 3	5 * 5 = 5 + 5 − 6 = 4

		$(m\ 5+1=\ 6)$	$(m\ 5+2>6)$	by (*ii*)	by (*ii*) $(m\ 5+4>6)$	by (*ii*)

Now the row headed by $*$ coincides with the row headed by 0.

$\therefore$ (e =) 0 $\in$ A is the identity element for the binary operation $*$ here.

(m $a * e = a = e * a \quad \forall\, a \in$ A)

Now we are to find inverse of each element $a \neq 0 \in$ A.

In the row headed by $a = 1$, identity 0 occurs at last place and the entry vertically above it is $5 = 6 - 1 = 6 - a$

m Inverse of $a = 1$ is $6 - a$ $(=5)$ $|a * b = e$

In the row headed by $a = 2$, identity 0 occurs at 5th place and the entry vertically above it is $4 = 6 - 2 = 6 - a$

$\therefore$ Inverse of $a = 2$ is $6 - a$ $(=4)$ $|a * b = e$

In the row headed by $a = 3$, identity 0 occurs at 4th place and the entry vertically above it is $3 = 6 - 3 = 6 - a$

In the row headed by $a = 4$, identity 0 occurs at third place and the entry vertically above it is $2 = 6 - 4 = 6 - a$ $(a * b = e)$

In the row headed by $a = 5$, identity 0 occurs at second place and the entry vertically above it is $1 = 6 - 5 = 6 - a$.

$\therefore$ Inverse of each $a \neq 0 \in$ A is $(6 - a) \in$ A.

Remarks: The reader is strongly suggested to do every question of binary operation on a finite set by the help of composition Table.

15. **Let A = {– 1, 0, 1, 2}, B = {– 4, – 2, 0, 2} and f, g : A $\to$ B be functions defined by $f(x) = x^2 - x$, $x \in$ A and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in$ A. Are f and g equal? Justify your answer.**

Sol. Given: Set A = {– 1, 0, 1, 2} and set B = {– 4, – 2, 0, 2}.

Function f : A $\to$ B is defined by $f(x) = x^2 - x$, $x \in$ A ...(*i*)

and g : A $\to$ B is defined by $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in$ A ...(*ii*)

We know that two functions f and g are said to be **equal** if f and g have same domain (which is same here namely set A) and $f(x) = g(x)$ for all $x \in$ A.

Here A = {– 1, 0, 1, 2}

From (*i*), $f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$

From (*ii*), $g(-1) = 2\left|-1 - \frac{1}{2}\right| - 1 = 2\left|-\frac{3}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1$

$= 3 - 1 = 2 \qquad \therefore f(-1) = g(-1)\ (= 2)$

From (i), $f(0) = 0^2 - 0 = 0 - 0 = 0$

From (ii), $g(0) = 2\left|-\frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\therefore$ $f(0) = g(0) = (= 0)$

From (i), $f(1) = 1^2 - 1 = 1 - 1 = 0$

From (ii), $g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

$\therefore$ $f(1) = g(1)\ (= 0)$

From (i), $f(2) = 2^2 - 2 = 4 - 2 = 2$

and $g(2) = 2\left|2 - \frac{1}{2}\right| - 1 = 2\left|\frac{3}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1$

$$= 3 - 1 = 2$$

$\therefore \quad f(2) = g(2) \ (= 2)$

$\therefore \quad f(x) = g(x)$ for all $x \in$ their common domain A.

$\therefore \quad f = g.$

16. Let A = {1, 2, 3}. Then number of relation containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4.

Sol. Given: Set A = {1, 2, 3}

$\therefore$ A × A = {1, 2, 3} × {1, 2, 3}

A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} ...(*i*)

This set A × A has 3 × 3 = 9 elements.

We know that every relation on the set A is a subset of A × A.

We know that a reflexive relation on the set A must contain all elements of the type $(a, a)\ \forall\ a \in$ A...(*ii*)

$\therefore$ A relation on this set A containing (1, 2) and (1, 3) (given) which is reflexive and symmetric but not transitive is

$$R = \{\underbrace{(1, 1), (2, 2), (3, 3)}_{\text{(for reflexive)}}, \underbrace{(1, 2)}_{\text{(given)}}, \underbrace{(2, 1)}_{\text{For symmetric}}, \underbrace{(1, 3)}_{\text{(given)}}, (3, 1)\} \quad ...(iii)$$

This relation R has 7 elements out of 9 elements of A × A.

Now element (2, 3) of A × A can't be included in R above because then (3, 2) will also have to be included in R.

(......R is symmetric (given))

and then R will become = A × A and hence will be an equivalence relation and hence will be transitive also.

The relation R given by (*iii*) is not transitive because (2, 1) ∈ R and (1, 3) ∈ R but (2, 3) ∉ R.

$\therefore$ Relation R given by (*ii*) is the only one required relation.

$\therefore$ Option (A) is the correct answer.

17. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4.

Sol. Given: Set A = {1, 2, 3}

$\therefore$ A × A = {1, 2, 3} × {1, 2, 3}

A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} ...(*i*)

This set A × A has 3 × 3 = 9 elements.

We know that every relation on the set A is a subset of A × A.

We know that a reflexive relation on the set A must contain all elements of the type $(a, a)\ \forall\ a \in$ A...(*ii*)

We know (and can observe from eqn. (*i*)) that A × A is the largest equivalence relation on the set A (of course containing (1, 2) also) ...(*iii*)

Let us write another equivalence relation R on the set A containing (1, 2).

This equivalence relation

$$R = \{\underbrace{(1, 1), (2, 2), (3, 3)}_{\text{(for reflexive)}}, \underbrace{(1, 2)}_{\text{(given)}}, \underbrace{(2, 1)}_{\text{For symmetric}}\} \quad \ldots\ldots\ldots\ldots (iv)$$

Now (1, 3) can't be included in R because then (3, 1) will also has to be taken in R ($\because$ R being equivalence relation is symmetric also) and then (3, 2) will also has to be taken in R.

($\because$ $(3, 1) \in R$, $(1, 2) \in R$) $\Rightarrow$ $(3, 2) \in R$ because R being an equivalence relation is transitive also)

Now if $(3, 2) \in R$, then (2, 3) has also to be included in R because R being equivalence relation is symmetric.

Hence relation R given by (*iv*) containing 5 elements of A × A will also include the remaining 4 elements of A × A namely (1, 3), (3, 1), (3, 2), (2, 3) and hence relation R given by (*iv*) will become A × A.

Therefore A × A and relation R given by (*iv*) are the only two equivalence relations on the set A containing (1, 2).

$\therefore$ Option (B) is the correct answer.

18. Let $f : R \to R$ be the Signum Function defined as

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and $g : R \to R$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then, does fog and gof coincide in $(0, 1]$?

Sol. **Given:** $f : R \to R$ defined as $f(x) = \begin{cases} 1, & x > 0 & ...(i) \\ 0, & x = 0 & ...(ii) \\ -1, & x < 0 & ...(iii) \end{cases}$

and $g : R \to R$ defined as $g(x) = [x]$ ...(*iv*)
where $[x]$ denotes the greatest integer less than or equal to x.
The given interval is $(0, 1]$.
On open interval $(0, 1)$;
$(fog)x = f(g(x)) = f([x])$ [By (*iv*)] $= f(0)$
($\because$ We know that on open interval $(0, 1)$, $[x] = 0$

$\therefore$ $(fog)x = 0 \ \forall \ x$ in open interval $(0, 1)$ ...(*v*)
Again $(fog)1 = f(g(1)) = f([1])$ [By (*iv*)]
$= f(1) = 1$ ($\because$ By (*i*), $f(x) = 1$ for $x > 0$) ...(*vi*)
Now on $(0, 1)$
$(gof)x = g(f(x)) = g(1)$ ($\because$ By (*i*), $f(x) = 1$ for $x > 0$) [By (*i*)]
$= [1]$ [By (*iv*)] $= 1$
$\therefore$ $(gof)x = 1$ for all x in $(0, 1]$ ...(*vii*)
The two functions *fog* and *gof* have the same domain $(0, 1]$ but $(fog)x$ (given by (*v*) and (*vi*) $\neq (gof)x$ (given by (*vii*)) $\forall \ x$ in $(0, 1]$
$\therefore fog \neq gof$ on $(0, 1]$ *i.e.,* the two functions don't coincide on $(0, 1]$.

19. Number of binary operations on the set $\{a, b\}$ is
(A) 10 **(B) 16** **(C) 20** **(D) 8.**

Sol. The given set $\{a, b\}$ (= A (say)) has 2 (= n) elements.
$\therefore$ Number of binary operations on this set A.

$$= n^{(n^2)} \text{ (Formula)} = 2^{(2^2)} = 2^4 = 16$$

$\therefore$ Option (B) is the correct answer.